

## Stability analysis of equilibria in nonlinear systems

Consider the  $n$ -dimensional nonlinear dynamical system

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \quad (1)$$

where  $\mathbf{x}(t) = [x_1(t) \cdots x_n(t)]^T$  is a vector of phase variables,  $\mathbf{f} : D \rightarrow \mathbb{R}^n$ , and  $D$  is a subset of  $\mathbb{R}^n$ . In this course note we study the behavior of the nonlinear system (1) in a neighborhood of a fixed point. As is well known, fixed points are solutions to the nonlinear system of algebraic equations

$$\mathbf{f}(\mathbf{x}^*) = \mathbf{0}. \quad (2)$$

To study the flow in a neighborhood of a fixed point  $\mathbf{x}^*$  we consider a local coordinate system centered at  $\mathbf{x}^*$ , i.e. we define the new phase variables

$$\boldsymbol{\eta}(t, \mathbf{x}_0) = \mathbf{X}(t, \mathbf{x}_0) - \mathbf{x}^*. \quad (3)$$

Assuming that the initial condition  $\mathbf{x}_0$  is sufficiently close to  $\mathbf{x}^*$  and that  $\mathbf{f}$  is sufficiently smooth, we expand

$$\mathbf{f}(\mathbf{X}(t, \mathbf{x}_0)) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\eta}(t, \mathbf{x}_0)) \quad (4)$$

in a neighborhood of  $\mathbf{x}^*$ , i.e., for small  $\boldsymbol{\eta}(t, \mathbf{x}_0)$ . This yields

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\eta}(t, \mathbf{x}_0)) = \underbrace{\mathbf{f}(\mathbf{x}^*)}_{=\mathbf{0}} + \mathbf{J}_f(\mathbf{x}^*)\boldsymbol{\eta}(t, \mathbf{x}_0) + \mathbf{g}(\boldsymbol{\eta}), \quad (5)$$

where

$$\mathbf{J}_f(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x}^*)}{\partial x_1} & \cdots & \frac{\partial f_1(\mathbf{x}^*)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x}^*)}{\partial x_1} & \cdots & \frac{\partial f_n(\mathbf{x}^*)}{\partial x_n} \end{bmatrix} \quad (6)$$

is the Jacobian<sup>1</sup> of  $\mathbf{f}(\mathbf{x})$  evaluated at the fixed point  $\mathbf{x}^*$ , and  $\mathbf{g}(\boldsymbol{\eta})$  is the reminder of the Taylor series at  $\mathbf{x}^*$ . Of course  $\mathbf{g}(\boldsymbol{\eta})$  depends on  $\mathbf{x}^*$ . Moreover,

$$\mathbf{g}(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad \mathbf{J}_g(\mathbf{x}^*) = \mathbf{0}. \quad (7)$$

These conditions imply that  $\boldsymbol{\eta} = \mathbf{0}$  is indeed a fixed point, and that that  $\mathbf{g}(\boldsymbol{\eta})$  is at least quadratic in  $\boldsymbol{\eta}$ . This allows us to write the nonlinear dynamical system (1) at  $\mathbf{x}^*$  as

$$\begin{cases} \frac{d\boldsymbol{\eta}}{dt} = \mathbf{J}_f(\mathbf{x}^*)\boldsymbol{\eta} + \mathbf{g}(\boldsymbol{\eta}) \\ \boldsymbol{\eta}(0, \mathbf{x}_0) = \mathbf{x}_0 - \mathbf{x}^* \end{cases} \quad (8)$$

Note that (8) is completely equivalent to (1), since we retained all nonlinearities. Such nonlinearities are responsible for the slight variations in the local phase portraits displayed in Figure 1.

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<sup>1</sup>The Jacobian of  $\mathbf{f}(\mathbf{x})$  is a matrix-valued function that takes in a function  $\mathbf{f}(\mathbf{x})$  and it returns a  $n \times n$  matrix-valued function. The entries of such Jacobian matrix are functions. Of course, if we evaluate the Jacobian of  $\mathbf{f}(\mathbf{x})$  at a specific point  $\mathbf{x}^*$  then we obtain a matrix with real entries (provided  $\mathbf{f}$  is real).

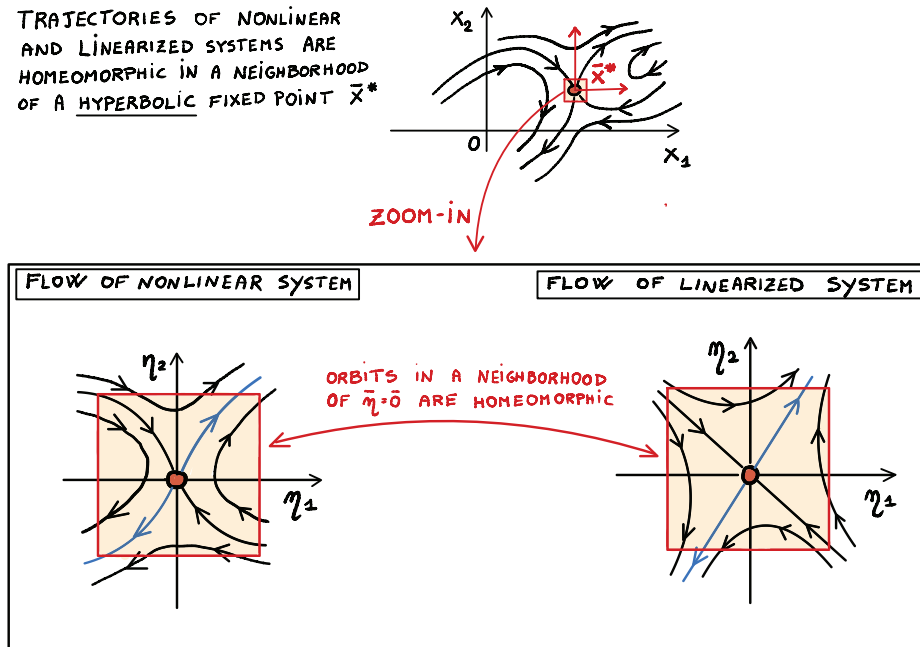


Figure 1: Geometric meaning of the Hartman-Grobman Theorem 1. The trajectories of a nonlinear system in a neighborhood of any hyperbolic fixed point are homeomorphic to the trajectories of the linearized system at  $\mathbf{x}^*$ . This means that the trajectories of the nonlinear and linearized system are not exactly the same in the neighborhood of  $\mathbf{x}^*$ , but they can be mapped to each other by a continuous transformation that has a continuous inverse. The reason why the trajectories are not the same can be traced back to the term  $\mathbf{g}(\boldsymbol{\eta})$  in equation (8).

**Theorem 1** (Hartman-Grobman). Let  $\mathbf{x}^* \in \mathbb{R}^n$  be a fixed point of the dynamical system (1). If the Jacobian (6) has no eigenvalue with zero real part then there exists a homeomorphism (i.e., continuous invertible mapping with continuous inverse) defined on some neighborhood of  $\mathbf{x}^*$  that takes orbits of the linear system  $\dot{\boldsymbol{\eta}} = \mathbf{J}_f(\mathbf{x}^*)\boldsymbol{\eta}$  and maps them into orbits of the system (8). The mapping preserves the orientation of the orbits.

This Theorem is stating that if  $\mathbf{x}^*$  is a hyperbolic<sup>2</sup> fixed point then the flow of the nonlinear dynamical system (8) is “homemorphic” (i.e., it can be mapped back and forth by a continuous nonlinear transformation) to the flow of the linearized system  $\dot{\boldsymbol{\eta}} = \mathbf{J}_f(\mathbf{x}^*)\boldsymbol{\eta}$ .

**Stable, unstable, and center subspaces.** In general, the eigenvalues of the Jacobian matrix  $\mathbf{J}_f(\mathbf{x}^*)$  and the associated subspaces can be grouped into three main classes (see Figure 2):

- **Stable subspace.** We denote the subspace spanned by the eigenvectors and the generalized eigenvectors associated with eigenvalues with negative real part as  $V^s$ . The subspace  $V^s$  is called *stable subspace* (or stable eigenspace if it is spanned by eigenvectors).
- **Unstable subspace.** We denote the subspace spanned by the eigenvectors and the generalized eigenvectors associated with eigenvalues with positive real part as  $V^u$ . The subspace  $V^u$  is called *unstable subspace* (or unstable eigenspace if it is spanned by eigenvectors).

<sup>2</sup>A fixed point  $\mathbf{x}^*$  is called *hyperbolic* if the Jacobian of  $\mathbf{J}_f(\mathbf{x}^*)$  has no eigenvalue with zero real part. Historically, the definition of hyperbolic fixed point stem from the fact that the orbits nearby a particular type of fixed point (saddle node) in two-dimensional non-dissipative systems resemble hyperbolas. This fails to hold in general.

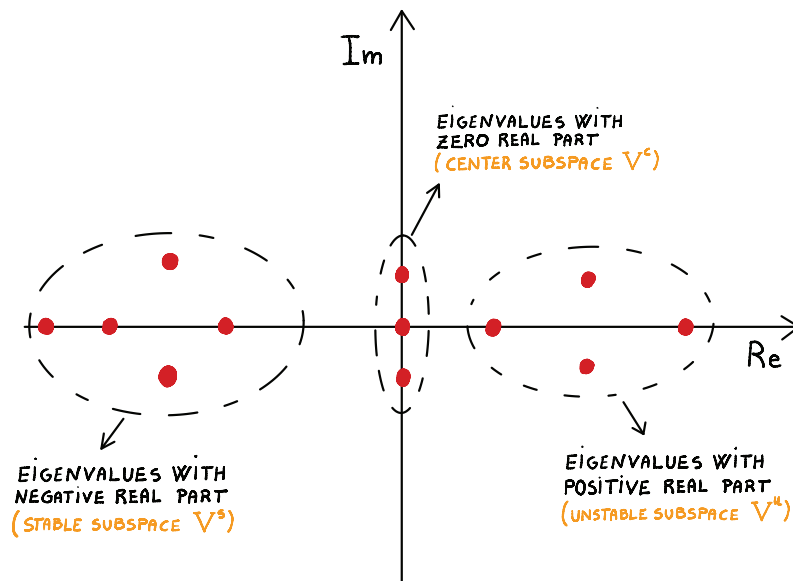


Figure 2: Eigenvalues of the Jacobian matrix  $\mathbf{J}_f(\mathbf{x}^*)$ , and definition of the associated subspaces.

- **Center subspace.** We denote the subspace spanned by the eigenvectors and the generalized eigenvectors associated with eigenvalues with zero real part as  $V^c$ . The subspace  $V^c$  is called *center subspace* (or center eigenspace if it is spanned by eigenvectors).

The Hartman-Grobman theorem applies to a fixed point  $\mathbf{x}^*$  with center subspace  $V^c$  reducing to just one element, i.e.,  $V^c = \{\mathbf{0}_{\mathbb{R}^n}\}$ . This means that  $\dim(V^c) = 0$ , i.e., the center subspace is zero dimensional. On the other hand, the *center manifold*<sup>3</sup> theorem discussed hereafter provides useful information on the stable, unstable, and center manifolds associated to a fixed point  $\mathbf{x}^*$ .

**Theorem 2** (Center manifold theorem). Let  $\mathbf{x}^* \in \mathbb{R}^n$  be a fixed point of the dynamical system (1), and let  $V^s$ ,  $V^u$  and  $V^c$  be the stable, unstable and center subspaces defined by (generalized) eigendecomposition of the Jacobian matrix  $\mathbf{J}_f(\mathbf{x}^*)$  defined in (6). Then there exist two unique stable and unstable invariant manifolds<sup>4</sup>  $W^s$  and  $W^u$  of the same dimension of  $V^s$  and  $V^u$  and tangential to  $V^s$  and  $V^c$  at  $\mathbf{x}^*$ , and a (not necessarily unique<sup>5</sup>) center manifold  $W^c$  of the same dimension of  $V^c$  and tangential to  $V^c$  at  $\mathbf{x}^*$ . If  $\mathbf{f}$  in (1) is of class  $C^k$  then  $W^s$  and  $W^u$  are of class  $C^k$ , while  $W^c$  is of class  $C^{k-1}$ .

It is useful to sketch the stable and unstable subspaces  $V^s$  and  $V^u$  together with the stable and stable manifolds  $W^s$  and  $W^u$  for 2D a saddle node and for a 2D stable node. In the latter case, the stable subspace has dimension 2, and therefore all curves in a neighborhood of  $\mathbf{x}^*$  are part of the stable manifold  $W^s$ .

**Stability analysis of hyperbolic fixed points in two-dimensional systems.** In this section we provide a few examples of stability analysis of a hyperbolic fixed point in two-dimensional nonlinear dynamical systems.

<sup>3</sup>A manifold can be thought of as a geometric object embedded in the Euclidean space  $\mathbb{R}^n$ . For example, a smooth (non-intersecting) curve in  $\mathbb{R}^2$  or a smooth surface in  $\mathbb{R}^3$  are examples of manifolds. More generally one can define a manifold as a space that is locally Euclidean.

<sup>4</sup>An invariant manifold  $W \subseteq \mathbb{R}^n$  is a manifold such that for all  $\mathbf{x}_0 \in W$  we have that  $X(t, \mathbf{x}_0) \in W$ .

<sup>5</sup>If  $\mathbf{f}(\mathbf{x})$  is  $C^\infty$  then it is possible to find a  $C^r$  center manifold for each  $r < \infty$ .

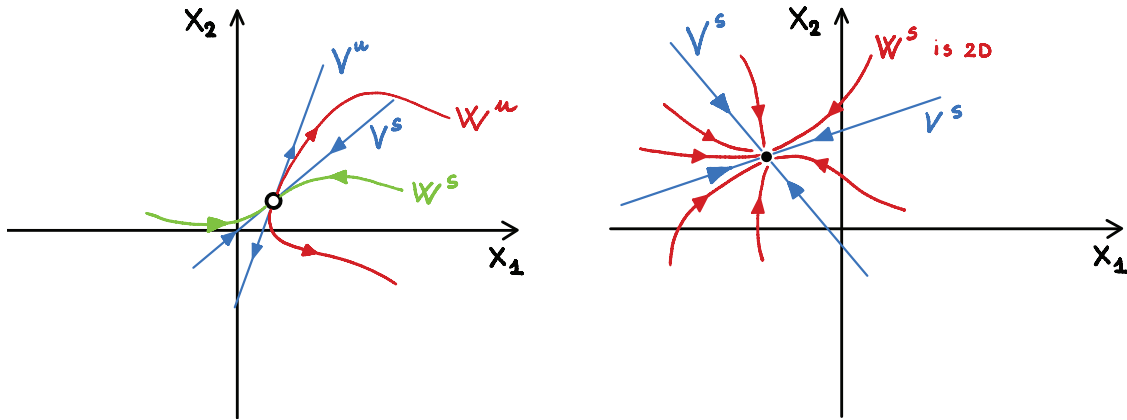


Figure 3: Stable and unstable eigenspaces  $V^s$  and  $V^u$ , and stable and unstable manifolds  $W^s$  and  $W^u$  of a two-dimensional saddle node and a two-dimensional stable node. Note that the stable and unstable manifolds of the saddle node are one-dimensional and tangent to the stable and unstable eigenspaces at fixed point. The stable eigenspace of the stable node is two-dimensional. Hence the the stable manifold is two-dimensional as well. Hence the tangency condition of  $W^s$  to  $V^s$  in this case reduces to the trivial statement that all trajectories belong to the stable manifold, at least locally.

**Stability analysis of hyperbolic fixed points.** Consider the following Volterra-Lotka model governing the population dynamics two interacting species competing for some common resource.

$$\begin{cases} \frac{dx_1}{dt} = x_1(3 - x_1 - 2x_2) \\ \frac{dx_2}{dt} = x_2(2 - x_1 - x_2) \end{cases} \tag{9}$$

The nullclines are

$$\dot{x}_1 = 0 \Rightarrow x_1 = 0, \quad x_2 = \frac{3}{2} - \frac{1}{2}x_1, \tag{10}$$

$$\dot{x}_2 = 0 \Rightarrow x_2 = 0, \quad x_2 = 2 - x_1. \tag{11}$$

Fixed points are located at the intersections of the nullclines. As shown in Figure 4 we obtain

$$\mathbf{x}_A^* = (0, 0), \quad \mathbf{x}_B^* = (0, 2), \quad \mathbf{x}_C^* = (1, 1), \quad \mathbf{x}_D^* = (3, 0). \tag{12}$$

The Jacobian of (9) is easily obtained as

$$\mathbf{J}_f(\mathbf{x}) = \begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -x_2 & 2 - x_1 - 2x_2 \end{bmatrix} \tag{13}$$

Let us study the flow of the nonlinear system in a neighborhood of the fixed point  $\mathbf{x}_C^* = (1, 1)$ . The Jacobian at  $\mathbf{x}_C^*$  is

$$\mathbf{J}_f(\mathbf{x}_C^*) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}, \tag{14}$$

and it has eigenvalues

$$\lambda_1 = -1 - \sqrt{2} < 0, \quad \lambda_2 = -1 + \sqrt{2} > 0. \tag{15}$$

Therefore the fixed point  $\mathbf{x}_C^*$  is hyperbolic (saddle node). The stable and unstable eigenspaces of the saddle node are spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \tag{16}$$



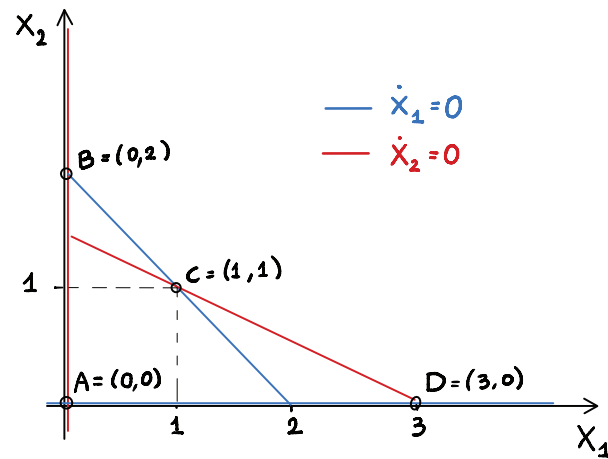


Figure 4: Fixed points of the Volterra-Lotka model (9).

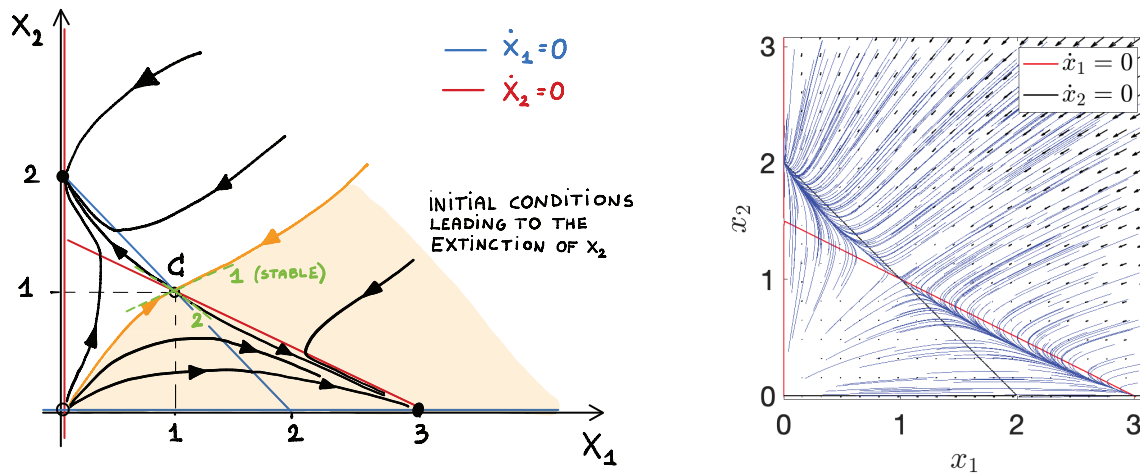


Figure 5: Phase portrait of the Volterra-Lotka model (9). The stable manifold of the saddle node determines which species is going to survive.

which are eigenvectors of (14) corresponding to  $\lambda_1$  and  $\lambda_2$ . Based on Theorem 2, the stable and unstable manifolds of the saddle node are tangent to the tangent eigenspaces stable and unstable manifolds are tangent to the eigendirections. Proceeding similarly for the other points, it is straightforward to find that  $\mathbf{x}_A^*$  is an unstable node, while  $\mathbf{x}_B^*$  and  $\mathbf{x}_D^*$  are stable nodes. In Figure 5 we sketch the phase portrait of the system, and compare it with a numerically computed portrait.

*Example:* Consider the nonlinear system

$$\begin{cases} \frac{dx_1}{dt} = 1 - (\mu + 1)x_1 + x_1^2 x_2 \\ \frac{dx_2}{dt} = \mu x_1 - x_1^2 x_2 \end{cases} \tag{17}$$

where  $\mu > 0$  is a real parameter. We allow  $\mu$  to vary<sup>6</sup>, since this will change the location of the fixed points

<sup>6</sup>By allowing  $\mu$  in (17) to vary, we are effectively studying potential bifurcations of the system, in particular bifurcations

and their stability properties. The nullclines are obtained by setting

$$\begin{cases} 1 - (\mu + 1)x_1 + x_1^2 x_2 = 0 \\ x_1(\mu - x_1 x_2) = 0 \end{cases} \Rightarrow \begin{cases} x_2 = \frac{\mu + 1}{x_1} - \frac{1}{x_1^2} \quad (\text{for } x_1 \neq 0) \\ x_1 = 0, \quad \text{or} \quad x_2 = \frac{\mu}{x_1} \end{cases}$$

The fixed points are at the intersections of the nullclines. By substituting  $x_2 = \mu/x_1$  into the equation defining the nullcline  $\dot{x}_1 = 0$  we obtain

$$\frac{\mu}{x_1} = \frac{\mu + 1}{x_1} - \frac{1}{x_1^2} \quad \Rightarrow \quad x_1^*(\mu) = 1. \quad (18)$$

Correspondingly,

$$\begin{aligned} x_2^*(\mu) &= \frac{\mu + 1}{x_1^*(\mu)} - \frac{1}{x_1^*(\mu)^2} \\ &= \mu + 1 - 1 \\ &= \mu. \end{aligned} \quad (19)$$

Therefore, we obtain the unique fixed point

$$(x_1^*(\mu), x_2^*(\mu)) = (1, \mu). \quad (20)$$

The Jacobian of the system (17) is

$$J_{\mathbf{f}}(x_1, x_2, \mu) = \begin{bmatrix} -(\mu + 1) + 2x_1 x_2 & x_1^2 \\ \mu - 2x_1 x_2 & -x_1^2 \end{bmatrix}. \quad (21)$$

The (linear) stability of the fixed point (20) is determined by the eigenvalues of

$$J_{\mathbf{f}}(x_1^*(\mu), x_2^*(\mu), \mu) = \begin{bmatrix} \mu - 1 & 1 \\ -\mu & -1 \end{bmatrix} \quad (22)$$

The associated characteristic polynomial

$$p(\lambda) = \lambda^2 - (\mu - 2)\lambda + 1 \quad (23)$$

has roots

$$\lambda_{1,2}(\mu) = \frac{(\mu - 2) \pm \sqrt{(\mu - 2)^2 - 4}}{2}. \quad (24)$$

In Figure 6 we plot the eigenvalues (24) as a function of  $\mu$ . Based on such eigenvalue analysis, it is seen that the fixed point (20) is:

- a stable spiral for  $0 < \mu < 2$ ;
- a non-hyperbolic fixed point for  $\mu = 2$ . Center manifold analysis outlined later in this course note allows us to conclude that the non-hyperbolic fixed point is a stable spiral;
- an unstable spiral for  $2 < \mu < 4$ ;
- an unstable degenerate node for  $\mu = 4$ ;
- a repeller for  $\mu > 4$ .

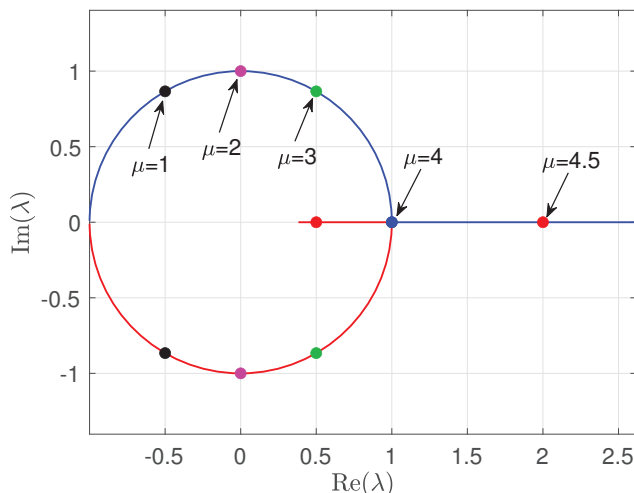


Figure 6: Eigenvalues of the Jacobian matrix (22) as a function of  $\mu$ .

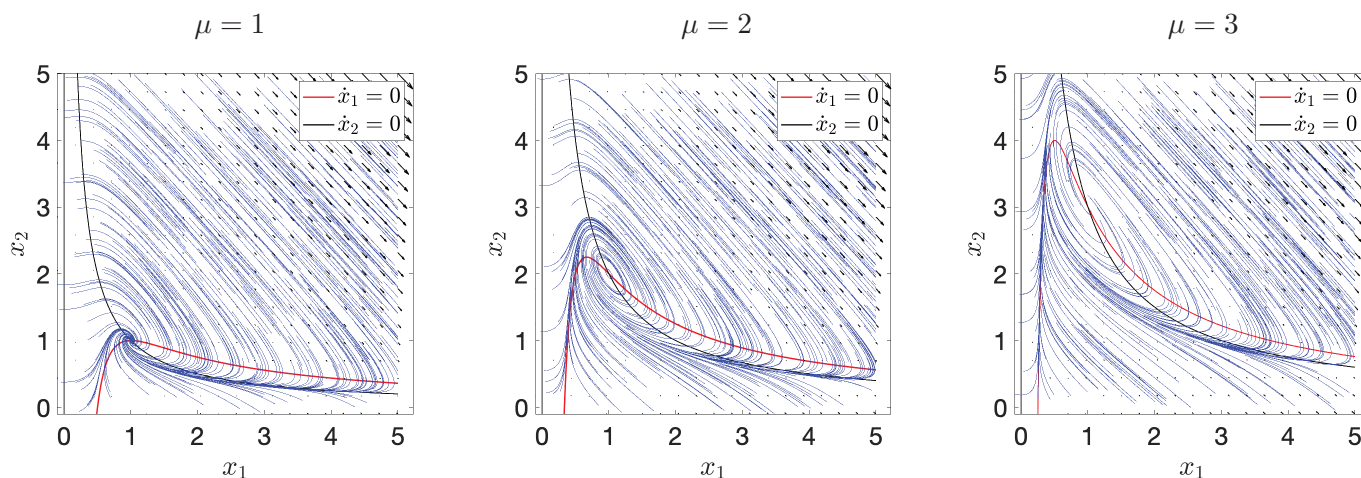


Figure 7: Phase portraits of (17) for different values of  $\mu$ .

For  $\mu = 2$  linear stability analysis predicts a center ( $\lambda_{1,2} = \pm i$ ). However, such fixed point is not hyperbolic and therefore such conclusion does not hold. Indeed the analysis of the center manifold outlined later in this course note allows us to conclude that for  $\mu = 2$  we have a stable spiral. For  $\mu = 4$  we have  $\lambda_{1,2} = 1$ . The geometric multiplicity of such eigenvalue is 1, and therefore at  $\mu = 4$  we have an unstable degenerate node. The phase portrait of the system is shown in Figure 7 for different values of  $\mu$ .

**Calculation of one-dimensional local center manifolds in two-dimensional systems.** Next, we study stability of non-hyperbolic fixed points in a two-dimensional dynamical system with one zero eigenvalue. Such stability can be studied by computing the dynamics on the center manifold  $W^c$  in a neighborhood of the fixed point  $\mathbf{x}^* \in \mathbb{R}^2$ . To this end, we represent such *local center manifold*  $W^c$  as a graph of a smooth function  $h$ , i.e.,

$$W^c = \{(x_1, x_2) \in \mathbb{R}^2 \text{ such that } x_2 = h(x_1) \text{ for all } x_1 \text{ in a neighborhood of } x_1^*\}. \quad (25)$$

According to the center manifold Theorem 2, there are three conditions that the function  $h(x_1)$  needs to satisfy in order to represent the center manifold in a neighborhood of the fixed point  $\mathbf{x}^*$ :

of equilibria.

1.  $(x_1, h(x_1))$  needs to pass through the fixed point, i.e.,

$$x_2^* = h(x_1^*) \quad (26)$$

2.  $h(x_1)$  needs to be tangent to  $V^c$  at the fixed point  $\mathbf{x}^*$ . This means that the slope  $h(x_1)$  must be the same as the slope<sup>7</sup> of  $V^c$  at  $x_1^*$ . Such slope is identified by the “center” eigenvector of  $\mathbf{J}_f(\mathbf{x}^*)$ .

3.  $W^c$  must be an invariant manifold. This means that any trajectory trajectory  $(x_1(t), x_2(t))$  on  $W^c$  must satisfy

$$x_2(t) = h(x_1(t)) \quad \Rightarrow \quad \frac{dx_2}{dt} = \frac{dh(x_1)}{dx_1} \frac{dx_1}{dt}, \quad (27)$$

i.e.,

$$f_2(x_1, h(x_1)) = \frac{dh(x_1)}{dx_1} f_1(x_1, h(x_1)). \quad (28)$$

These three conditions allow us to determine a power series expansion of the (one-dimensional) center manifold  $W^c$  in a neighborhood of the fixed point  $\mathbf{x}^*$ . Let's see some examples.

*Example:* Consider the nonlinear system

$$\begin{cases} \frac{dx_1}{dt} = x_1 x_2 \\ \frac{dx_2}{dt} = -x_2 - x_1^2 \end{cases} \quad (29)$$

The nullclines are

$$\dot{x}_1 = 0 \quad \Leftrightarrow \quad x_1 = 0 \quad \text{or} \quad x_2 = 0, \quad (30)$$

$$\dot{x}_2 = 0 \quad \Leftrightarrow \quad x_2 = -x_1^2. \quad (31)$$

Hence, there exists only one fixed point at the intersection of the nullclines which is

$$\mathbf{x}^* = (0, 0). \quad (32)$$

The Jacobian of the system (29) is

$$\mathbf{J}_f(\mathbf{x}) = \begin{bmatrix} x_2 & x_1 \\ -2x_1 & -1 \end{bmatrix}. \quad (33)$$

By evaluating  $\mathbf{J}_f(\mathbf{x})$  at the fixed point  $\mathbf{x}^* = (0, 0)$  we obtain

$$\mathbf{J}_f(\mathbf{0}) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \quad (34)$$

The eigenvalues of  $\mathbf{J}_f(\mathbf{0})$  are

$$\lambda_c = 0 \quad \text{and} \quad \lambda_s = -1. \quad (35)$$

Correspondingly, we have a center eigenspace  $V^c$  and a stable eigenspace  $V^s$ , both of dimension one. Such eigenspaces are spanned by the eigenvectors

$$\mathbf{v}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (36)$$

In Figure 8 we sketch the nullclines and the eigenspaces  $V^c$  and  $V^s$ . Next, we compute the local center

<sup>7</sup>If the center subspace  $V^s$  is a vertical line then we need to compute a preliminary coordinate transformation, e.g., use the so-called normal coordinates.

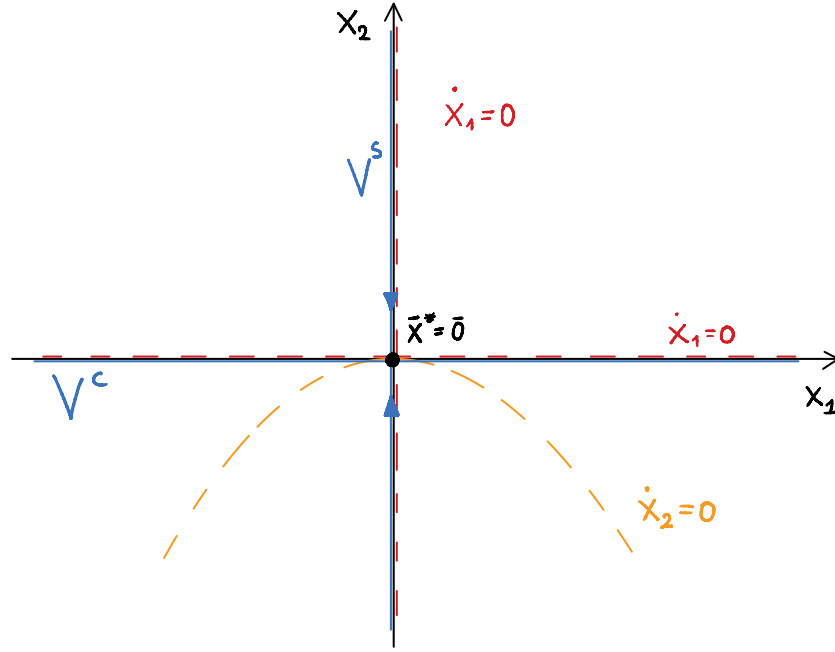


Figure 8: Nonlinear system (29). Stable ( $V^s$ ) and center ( $V^c$ ) eigenspaces associated with the fixed point  $\mathbf{x}^* = (0, 0)$ .

manifold  $W^c$  in a neighborhood of the fixed point  $\mathbf{x}^* = (0, 0)$ . To this end, we consider the following power series expansion of the function  $h(x_1)$  appearing in (25)

$$x_2 = h(x_1) = a + bx_1 + cx_1^2 + dx_1^3 + \dots, \quad (37)$$

where  $a, b, c$ , etc. are coefficients to be determined. By enforcing that  $W^c$  passes through the fixed point  $(0, 0)$  and is tangent to  $V^c$  at  $(0, 0)$  we obtain

$$\begin{cases} 0 = h(0) = a & \Leftrightarrow & a = 0 \\ 0 = h'(0) = b & \Leftrightarrow & b = 0 \end{cases} \quad (38)$$

Therefore we are left with

$$h(x_1) = cx_1^2 + dx_1^3 + ex_1^4 + \dots \quad (39)$$

At this point we impose that the dynamics on the local center manifold  $W^c$  is invariant, which means that any trajectory with initial condition on  $W^c$  stays on  $W^c$ . This condition is expressed mathematically by equation (28), which can be written the system (29) as

$$-h(x_1) - x_1^2 = \underbrace{(2cx_1 + 3dx_1^2 + \dots)}_{h'(x_1)} x_1 h(x_1). \quad (40)$$

Substituting  $h(x_1)$  yields

$$-(cx_1^2 + dx_1^3 + ex_1^4 \dots) - x_1^2 = (2cx_1 + 3dx_1^2 + \dots) x_1 (cx_1^2 + dx_1^3 + \dots), \quad (41)$$

i.e.,

$$-(c+1)x_1^2 - dx_1^3 - ex_1^4 + \dots = 2c^2x_1^4 + 5cdx_1^5 + \dots \quad (42)$$

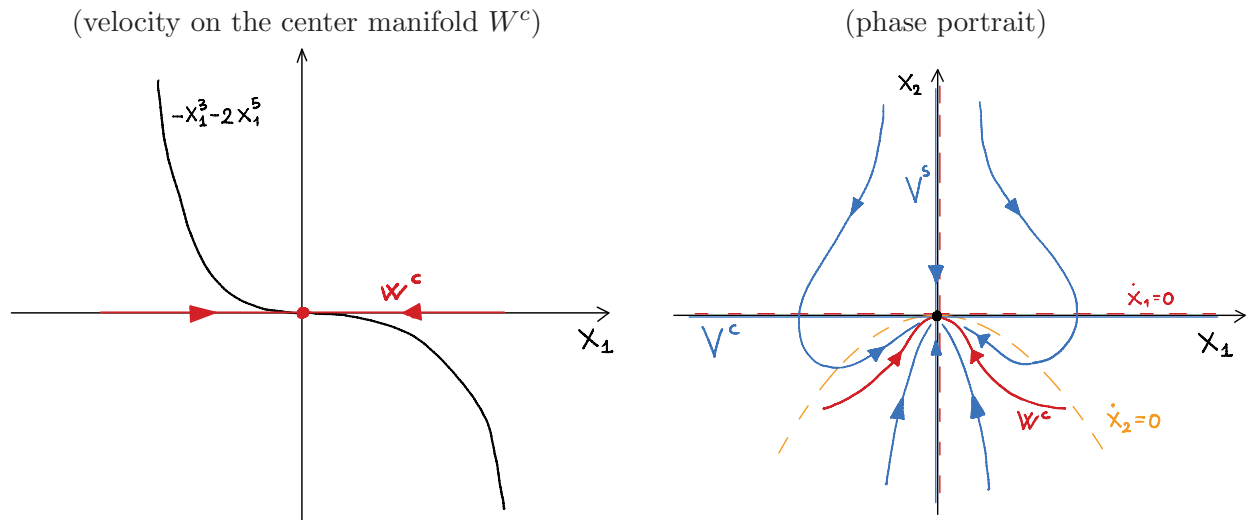


Figure 9: Nonlinear system (29). Local center manifold  $W^c$  at the non-hyperbolic fixed point  $(0, 0)$ .

Since we are free to choose  $x_1$  as small as we like, the previous equation yields the following conditions (match the coefficients multiplying the same power of  $x_1$  at the left and the right hand sides)

$$c + 1 = 0, \quad d = 0, \quad -e = 2c^2, \tag{43}$$

i.e.,

$$c = -1, \quad d = 0, \quad e = -2. \tag{44}$$

This yields the following power series expansion of the local center manifold  $W^c$

$$x_2 = h(x_1) = -x_1^2 - 2x_1^4 + \dots \tag{45}$$

The dynamics on this manifold can be obtained by substituting  $x_2 = h(x_1)$  into the first equation of the system (29). This yields

$$\frac{dx_1}{dt} = -x_1^3 - 2x_1^5 + \dots \tag{46}$$

Hence  $\dot{x}_1$  always points towards the origin when evaluated along the manifold  $W^c$ , i.e.,  $W^c$  is *stable* (see Figure 9). In Figure 10 we plot the phase portrait of (29) computed numerically.

*Example:* Let us provide another example of analysis of a two-dimensional non-hyperbolic fixed point. To this end, consider the nonlinear system

$$\begin{cases} \frac{dx_1}{dt} = -x_1x_2 \\ \frac{dx_2}{dt} = x_1 - x_2 \end{cases} \tag{47}$$

The nullclines are

$$\dot{x}_1 = 0 \Leftrightarrow x_1 = 0 \quad \text{or} \quad x_2 = 0, \tag{48}$$

$$\dot{x}_2 = 0 \Leftrightarrow x_2 = x_1. \tag{49}$$

Hence, there exists only one fixed point at

$$\mathbf{x}^* = (0, 0). \tag{50}$$

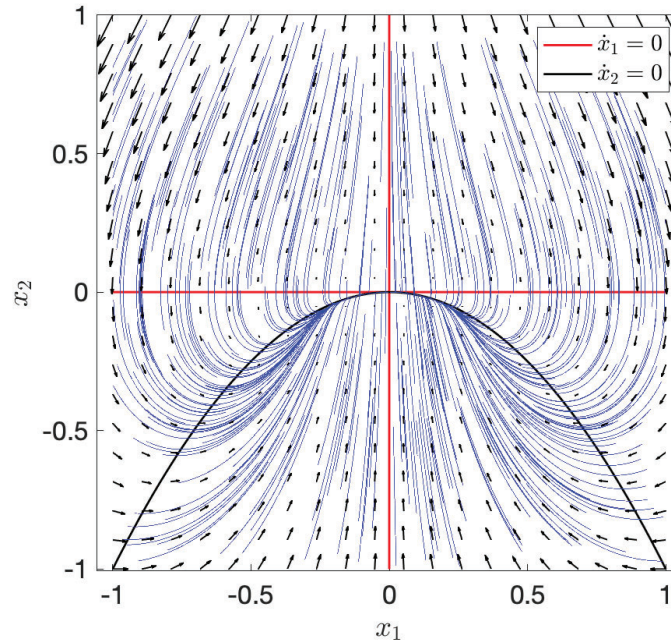


Figure 10: Phase portrait of the dynamical system (29). Note that the numerical results indicate that there may be an infinite number of center manifolds at  $\mathbf{x}^* = (0, 0)$  (all curves passing through  $(0, 0)$  with horizontal tangent at  $(0, 0)$ ). However, the Taylor series expansions of any two center manifolds at  $(0, 0)$  agree to all orders.

The Jacobian of the system (47) is

$$\mathbf{J}_f(\mathbf{x}) = \begin{bmatrix} -x_2 & -x_1 \\ 1 & -1 \end{bmatrix} \quad (51)$$

By evaluating  $\mathbf{J}_f(\mathbf{x})$  at the fixed point  $\mathbf{x}^* = (0, 0)$  we obtain

$$\mathbf{J}_f(\mathbf{0}) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}. \quad (52)$$

The eigenvalues of  $\mathbf{J}_f(\mathbf{0})$  are

$$\lambda_c = 0 \quad \text{and} \quad \lambda_s = -1. \quad (53)$$

Correspondingly we have a center eigenspace  $V^c$  and a stable eigenspace  $V^s$ , both of dimension one. Such eigenspaces are spanned by the eigenvectors

$$\mathbf{v}_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (54)$$

To study stability of the non-hyperbolic fixed point  $\mathbf{x}^* = (0, 0)$ , we compute the local center manifold  $W^c$  at  $\mathbf{x}^*$ . Based on Theorem 2,  $W^c$  is a  $C^\infty$  one-dimensional manifold and therefore it can be represented locally as a graph of a  $C^\infty$  one-dimensional function  $h$  as

$$x_2 = h(x_1). \quad (55)$$

The function  $h$  must satisfies the conditions

$$\begin{cases} h(0) = 0 & W^c \text{ passes through the fixed point } \mathbf{x}^* = (0, 0), \\ h'(0) = 1 & W^c \text{ is tangent to } V^c \text{ at the fixed point } \mathbf{x}^* = (0, 0). \end{cases} \quad (56)$$

Expanding  $h(x_1)$  in a power series at  $\mathbf{x}^* = (0, 0)$  yields

$$h(x_1) = a + bx_1 + cx_1^2 + dx_1^3 + \dots \quad (57)$$

By enforcing conditions (56) we obtain

$$a = 0, \quad b = 1. \quad (58)$$

Hence,

$$h(x_1) = x_1 + cx_1^2 + dx_1^3 + \dots \quad (59)$$

As before, the other coefficients can be obtained by imposing that  $W^c$  is an invariant manifold, i.e., that trajectories starting in  $W^c$  stay in  $W^c$ . This is equivalent to imposing that the dynamical system (47) has (55) as trajectory, i.e.,

$$x_2(t) = h(x_1(t)) \quad \text{for all } t \geq 0, \quad (60)$$

where  $(x_1(t), x_2(t))$  is a solution of (47). Differentiating (60) with respect to time yields and using (47) yields

$$x_1 - h(x_1) = -\frac{dh(x_1)}{dx_1}x_1h(x_1). \quad (61)$$

Substituting the power series (59) into the previous equation we obtain

$$x_1 - x_1 - cx_1^2 - dx_1^3 - \dots = -x_1(1 + 2cx_1 + 3dx_1^2 + \dots)(x_1 + cx_1^2 + dx_1^3 + \dots), \quad (62)$$

i.e.,

$$-cx_1^2 - dx_1^3 - \dots = -x_1^2 - 3cx_1^3 + \dots \Rightarrow c = 1, \quad d = 3. \quad (63)$$

Hence, the power series expansion of the center manifold  $W^c$  in a neighborhood of  $\mathbf{x}^* = (0, 0)$  is

$$x_2 = h(x_1) = x_1 + x_1^2 + 3x_1^3 + \dots \quad (64)$$

The dynamics on the manifold  $W^c$  is obtained by substituting (60) into (47). This yields

$$\dot{x}_1 = -x_1(x_1 + x_1^2 + 3x_1^3 + \dots) = -x_1^2 - x_1^3 - 3x_1^4 + \dots \quad (65)$$

The right hand side suggests of this equation that the  $x_1$  component of the velocity on the center manifold  $W^c$  always points left (see Figure 11). Hence the fixed point  $(0, 0)$  is unstable. In Figure 12 we plot the phase portrait of (29) computed numerically.

**Non-uniqueness of center manifolds.** We've mentioned in Theorem 2 that center manifolds need not be unique. This can be seen from the following simple example. Consider the dynamical system

$$\begin{cases} \frac{dx_1}{dt} = x_1^2 \\ \frac{dx_2}{dt} = -x_2 \end{cases} \quad (66)$$

clearly,  $(x_1, x_2) = (0, 0)$  is a fixed point. The stable manifold  $W^s$  is the vertical axis  $x_1 = 0$ . Moreover,  $x_2 = 0$  is an invariant center manifold, but there are other center manifolds. In fact, eliminating  $t$  as the independent variable in (66), we obtain (for  $x_1 \neq 0$ )

$$\frac{dx_2}{dx_1} = -\frac{x_2}{x_1^2} \Rightarrow x_2(x_1) = \beta e^{1/x_1} \quad \beta \in \mathbb{R}. \quad (67)$$

Thus, the curves given by

$$h(x_1) = \begin{cases} \beta e^{1/x_1} & x_1 < 0 \\ 0 & x_1 \geq 0 \end{cases} \quad (68)$$



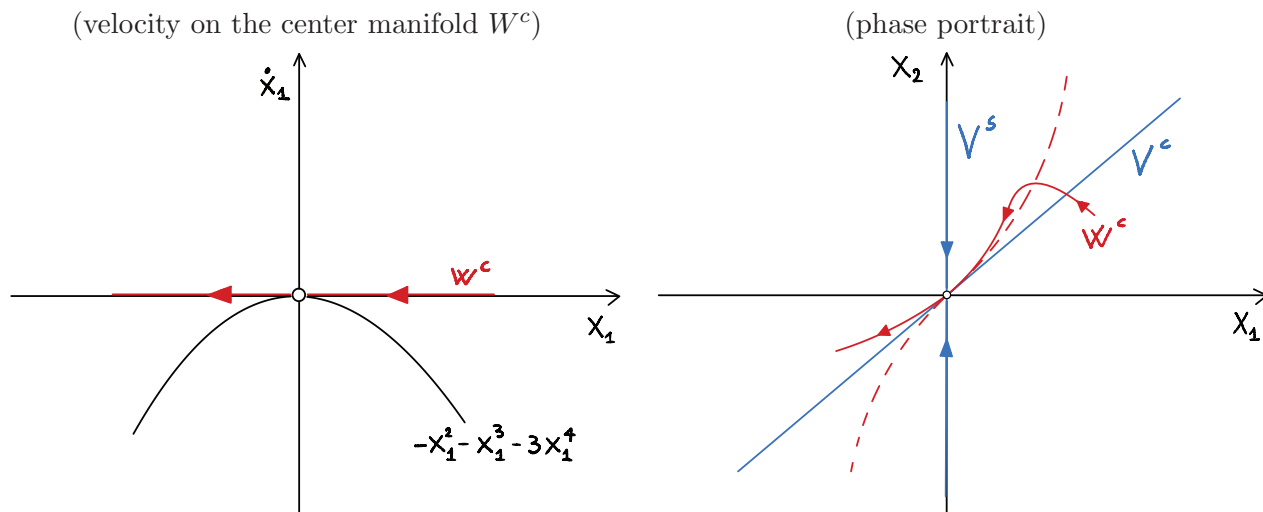


Figure 11: Nonlinear system (47). Stable and center eigenspaces  $V^s$  and  $V^c$ , and local center manifold  $W^c$  at the non-hyperbolic fixed point  $(0, 0)$ .

are a one-parameter (parametrized by  $\beta$ ) family of center manifolds of  $(x_1, x_2) = (0, 0)$ . These center manifolds are shown in Figure 13. It is easy to verify indeed that  $x_2(t) = \beta e^{1/x_1(t)}$  is an invariant manifold for the system (66). Moreover it is tangent to  $V^c$  ( $x_1$  axis), and it passes through  $(0, 0)$  (for  $x_1 \rightarrow 0^-$ ).

This example immediately brings up the following question: *In approximating the local center manifold via power series expansions, which center manifold is actually being approximated?* It can be shown that any two center manifolds of a given fixed point differ by (at most) transcendentally small terms. Thus, the Taylor series expansions of any two center manifolds at a given fixed point agree to all orders. Moreover, it can be shown that for an analytical system, if the series expansion of  $h$  converges, then there exists a unique analytical center manifold.

**Two-dimensional center manifolds.** Let us consider the case where the Jacobian matrix  $\mathbf{J}_f(\mathbf{x}^*)$  in (8) has two imaginary (complex conjugate) eigenvalues, i.e.,

$$\lambda_1 = i\omega \quad \lambda_2 = -i\omega, \quad (69)$$

where  $\omega$  is a nonzero real number. In Appendix A we show that the real Jordan form of  $\mathbf{J}_f(\mathbf{x}^*)$  is

$$\mathbf{A} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}. \quad (70)$$

Such real Jordan form is obtained by a real similarity transformation  $\mathbf{P}$  that has the real and the imaginary part of one eigenvector as columns. By defining new variables

$$\mathbf{q} = \mathbf{P}^{-1}\boldsymbol{\eta} \quad (71)$$

it is straightforward to transform the dynamical system (8) to

$$\begin{cases} \frac{dq_1}{dt} = \omega q_2 + H_1(q_1, q_2) \\ \frac{dq_2}{dt} = -\omega q_1 + H_2(q_1, q_2) \end{cases} \quad (72)$$

To study stability of the fixed point  $\mathbf{x}^*$ , we need to study the orbits of the nonlinear dynamical system (72) nearby  $\mathbf{q} = \mathbf{0}$ . A rather lengthy calculation establishes the local equivalency of (72) to the following

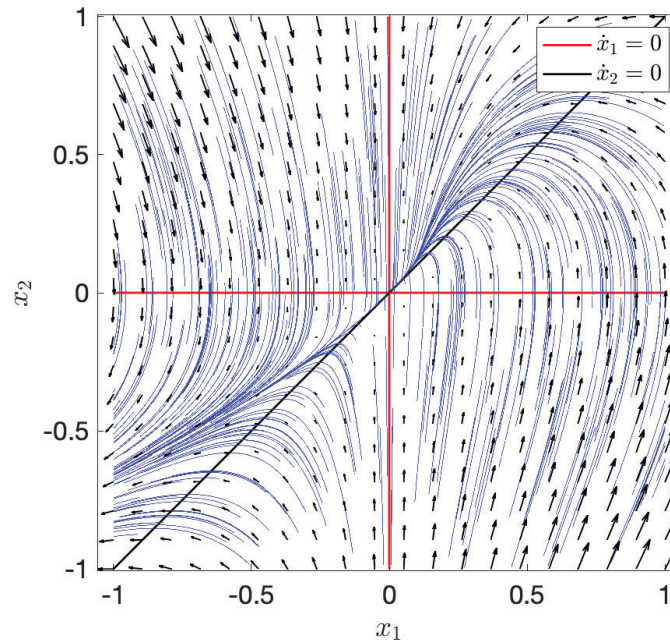


Figure 12: Phase portrait of the dynamical system (47).

dynamical system in polar coordinates ( $r$  and  $\theta$  are radius and angle of the phase vector with components  $(q_1, q_2)$ )

$$\begin{cases} \frac{dr}{dt} = ar^3 \\ \frac{d\theta}{dt} = -\omega + br^2 \end{cases} \quad (73)$$

where  $a$  a suitable constant. Therefore the trajectories nearby the fixed point fixed point  $\mathbf{x}^*$  are either spirals or centers, depending on the parameter  $a$ . It can be shown (see, e.g., the book by Guckenheimer and Holmes, “Nonlinear oscillations, dynamical systems and bifurcations of vector fields”, p. 154) that

$$\begin{aligned} a = & \frac{1}{16} \left[ \frac{\partial^3 H_1}{\partial q_1^3} + \frac{\partial^3 H_1}{\partial q_1 \partial q_2^2} + \frac{\partial^3 H_2}{\partial q_1^2 \partial q_2} + \frac{\partial^3 H_2}{\partial q_2^3} \right] + \\ & \frac{1}{16\omega} \left[ \frac{\partial^2 H_1}{\partial q_1 \partial q_2} \left( \frac{\partial^2 H_1}{\partial q_1^2} + \frac{\partial^2 H_1}{\partial q_2^2} \right) - \frac{\partial^2 H_2}{\partial q_1 \partial q_2} \left( \frac{\partial^2 H_2}{\partial q_1^2} + \frac{\partial^2 H_2}{\partial q_2^2} \right) - \right. \\ & \left. \frac{\partial^2 H_1}{\partial q_1^2} \frac{\partial^2 H_2}{\partial q_1^2} + \frac{\partial^2 H_1}{\partial q_2^2} \frac{\partial^2 H_2}{\partial q_2^2} \right], \end{aligned} \quad (74)$$

where all derivatives of  $H_1(\eta_1, \eta_2)$  and  $H_2(\eta_1, \eta_2)$  are evaluated at  $(0, 0)$ . Hence, if  $a < 0$  we get a stable spiral and if  $a > 0$  we get an unstable spiral. The case  $a = 0$  requires higher order Taylor expansions.

*Example:* Consider the dynamical system

$$\begin{cases} \frac{dx_1}{dt} = -x_2 - (x_1^2 + x_2^2) + x_1 x_2 \\ \frac{dx_2}{dt} = x_1 - (x_1^2 + x_2^2) - x_1 x_2 \end{cases} \quad (75)$$

The system has a fixed point at  $\mathbf{x}^* = (0, 0)$ . The Jacobian of (75) at  $(0, 0)$  is

$$\mathbf{J}_f(\mathbf{x}) = \begin{bmatrix} -2x_1 + x_2 & -1 - 2x_2 + x_1 \\ 1 - 2x_1 - x_2 & -2x_2 - x_1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{J}_f(0, 0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (76)$$

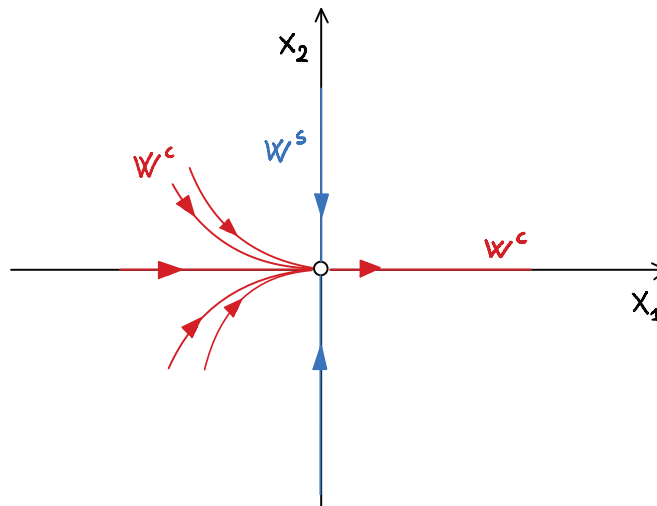


Figure 13: Non-uniqueness of center manifold for the fixed point  $\mathbf{x}^* = (0, 0)$  of the dynamical system (66).

The eigenvalues of  $\mathbf{J}_f(0, 0)$  are

$$\lambda_{1,2} = \pm i. \quad (77)$$

Hence,  $\mathbf{x}^* = (0, 0)$  is a *non-hyperbolic* fixed point with an associated two-dimensional center manifold. To study the dynamics nearby  $\mathbf{x}^* = (0, 0)$  we use the normal form (73) and calculate the coefficient (74) for

$$H_1(x_1, x_2) = -(x_1^2 + x_2^2) + x_1x_2 \quad H_2(x_1, x_2) = -(x_1^2 + x_2^2) - x_1x_2 \quad (78)$$

Note that in this case  $\omega$  is equal to one (compare (75) and (72)) and the third derivatives of  $(H_1, H_2)$  are both equal to zero. Moreover,

$$\frac{\partial^2 H_1}{\partial x_1 x_2} = 1, \quad \frac{\partial^2 H_2}{\partial x_1 x_2} = -1, \quad \frac{\partial^2 H_i}{\partial x_j^2} = -2, \quad (i, j = 1, 2). \quad (79)$$

Substituting these derivatives in (74) we yields

$$\begin{aligned} a &= \frac{1}{16} [1 \times (-2 - 2) - (-1) \times (-2 - 2) - (-2) \times (-2) + (-2) \times (-2)] \\ &= \frac{1}{16} [-4 - 4 - 4 + 4] \\ &= -\frac{1}{2} \end{aligned} \quad (80)$$

Hence, we conclude that the non-hyperbolic fixed point  $(0, 0)$  is a *stable spiral*. The phase portrait for this system is shown in Figure 14. Note that the stable spiral is enclosed by a *homoclinic orbit*, i.e., a trajectory that connects the unstable manifold and the stable manifold of the saddle node located nearby the spiral.

**Normal form of nonlinear dynamical systems at fixed points.** The center manifold Theorem 2 allows us to write any dynamical system in a neighborhood of an equilibrium point in a “normal form”. Such normal form differs from a standard linearization in that the dynamics on the subspace  $V^c$  is nonlinear. To obtain such normal form let us start from the nonlinear system (8), which represents (1) at the fixed point  $\mathbf{x}^*$ . We group the eigenvalues of the Jacobian  $\mathbf{J}(\mathbf{x}^*)$  as in Figure 2, and denote by

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & & \\ & \mathbf{B} & \\ & & \mathbf{C} \end{bmatrix} \quad (81)$$

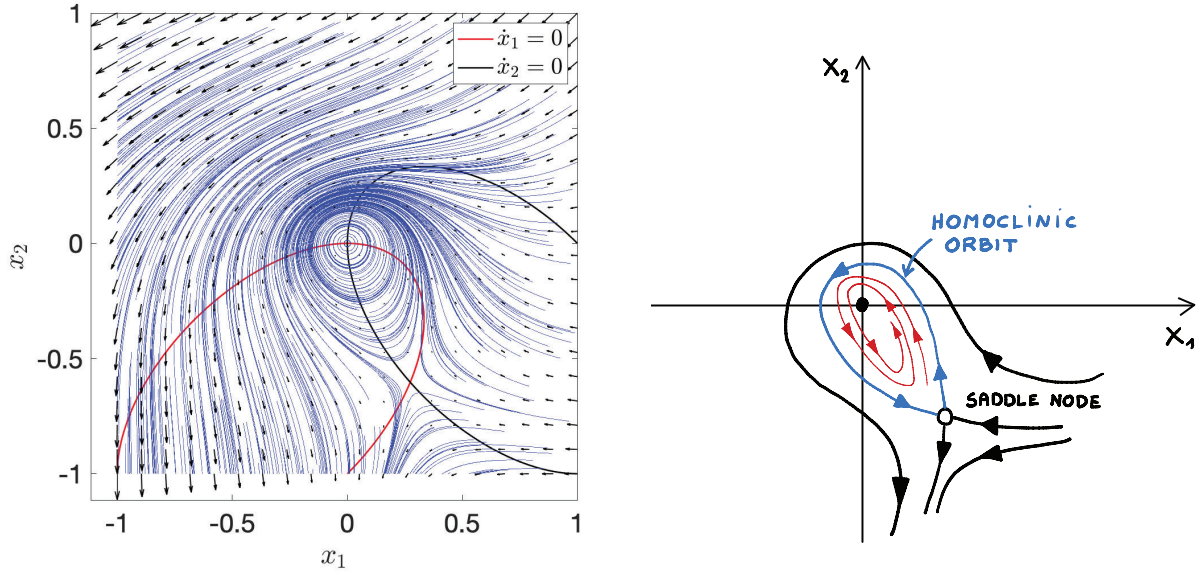


Figure 14: Phase portrait of the system (75). The system has a non-hyperbolic fixed point at  $\mathbf{x}^* = (0, 0)$ , which turns out to be a stable spiral. The stable spiral is enclosed by a *homoclinic trajectory*, i.e., an trajectory that connect the unstable manifold and the stable manifold of the saddle node that is located nearby.

The Jordan form of the Jacobian matrix  $\mathbf{J}(\mathbf{x}^*)$ . The projection matrix  $\mathbf{P}$  is

$$\mathbf{P} = [\mathbf{P}_c \quad \mathbf{P}_s \quad \mathbf{P}_u] \quad (82)$$

where  $\mathbf{P}_c$ ,  $\mathbf{P}_s$  and  $\mathbf{P}_u$  are projection matrices onto  $V^c$ ,  $V^s$  and  $V^u$ . Such projection matrices are made of generalized eigenvectors (columnwise) spanning each of the subspaces  $V^c$ ,  $V^s$  and  $V^u$ . The Jordan factorization of  $\mathbf{J}(\mathbf{x}^*)$  takes the form

$$\mathbf{J}(\mathbf{x}^*) = \mathbf{P}\mathbf{K}\mathbf{P}^{-1}. \quad (83)$$

Next, define a new set of variables

$$\mathbf{q} = \mathbf{P}^{-1}\boldsymbol{\eta}. \quad (84)$$

A substitution of (83) and (84) into (8) yields

$$\frac{d\mathbf{q}}{dt} = \mathbf{K}\mathbf{q} + \mathbf{P}^{-1}\mathbf{g}(\mathbf{P}\mathbf{q}). \quad (85)$$

Upon definition of

$$\mathbf{q} = \begin{bmatrix} \mathbf{c} \\ \mathbf{s} \\ \mathbf{u} \end{bmatrix} \quad (86)$$

this system can be split as

$$\begin{cases} \frac{d\mathbf{c}}{dt} = \mathbf{A}\mathbf{c} + \mathbf{f}_c(\mathbf{c}, \mathbf{s}, \mathbf{u}) & \text{dynamics in } V^c \text{ (}\mathbf{A} \text{ has eigenvalues with zero real part)} \\ \frac{d\mathbf{s}}{dt} = \mathbf{B}\mathbf{s} + \mathbf{f}_s(\mathbf{c}, \mathbf{s}, \mathbf{u}) & \text{dynamics in } V^s \text{ (}\mathbf{B} \text{ has eigenvalues with negative real part)} \\ \frac{d\mathbf{u}}{dt} = \mathbf{C}\mathbf{u} + \mathbf{f}_u(\mathbf{c}, \mathbf{s}, \mathbf{u}) & \text{dynamics in } V^u \text{ (}\mathbf{C} \text{ has eigenvalues with positive real part)} \end{cases} \quad (87)$$

If  $\|\mathbf{q}\|$  is very small then the nonlinear terms  $\mathbf{f}_s$  and  $\mathbf{f}_u$  are negligible with respect to  $\mathbf{B}\mathbf{s}$  and  $\mathbf{C}\mathbf{u}$ , respectively. This leaves us with the system

$$\begin{cases} \frac{d\mathbf{c}}{dt} = \mathbf{A}\mathbf{c} + \mathbf{f}_c(\mathbf{c}, \mathbf{s}, \mathbf{u}) \\ \frac{d\mathbf{s}}{dt} = \mathbf{B}\mathbf{s} \\ \frac{d\mathbf{u}}{dt} = \mathbf{C}\mathbf{u} \end{cases} \quad (88)$$

By using the center manifold theorem we can express the dynamics on  $W^c$  as a vector map

$$W^c = \{(\mathbf{c}, \mathbf{s}, \mathbf{u}) \in \mathbb{R}^n : \mathbf{s} = \mathbf{h}_s(\mathbf{c}) \quad \text{and} \quad \mathbf{u} = \mathbf{h}_u(\mathbf{c})\} \quad (89)$$

subject to the conditions

$$\begin{aligned} \mathbf{h}_s(\mathbf{0}) = \mathbf{0}, & \quad \mathbf{h}_u(\mathbf{0}) = \mathbf{0}, & (W^c \text{ passes through } \boldsymbol{\eta} = \mathbf{0}), \\ \nabla \mathbf{h}_s(\mathbf{0}) = \mathbf{0}, & \quad \nabla \mathbf{h}_u(\mathbf{0}) = \mathbf{0}, & (W^c \text{ is tangent to } V^s \text{ at } \boldsymbol{\eta} = \mathbf{0}). \end{aligned} \quad (90)$$

With the center manifold (89) available, we can decouple the system (88) as

$$\begin{cases} \frac{d\mathbf{c}}{dt} = \mathbf{A}\mathbf{c} + \mathbf{f}_c(\mathbf{c}, \mathbf{h}_s(\mathbf{c}), \mathbf{h}_u(\mathbf{c})) \\ \frac{d\mathbf{s}}{dt} = \mathbf{B}\mathbf{s} \\ \frac{d\mathbf{u}}{dt} = \mathbf{C}\mathbf{u} \end{cases} \quad (91)$$

This system of equations represents the generalization of the Hartman-Grobman theorem for non-hyperbolic fixed points. From (91) we see that the dynamics on the stable and stable subspaces of are trivial in normal coordinates, while the dynamics on the center manifold is essentially nonlinear.

## Appendix A: Real Jordan form of a 2D matrix with imaginary eigenvalues

In this Appendix we briefly describe the procedure to compute the real Jordan form of a  $2 \times 2$  matrix with complex conjugate eigenvalues. The generalization to  $n \times n$  matrices with real and complex conjugate eigenvalues is straightforward and can be built based the technique discussed hereafter and in the Appendix A of the course note 4. Let us illustrate how to compute the real Jordan form of a  $2 \times 2$  matrix using a simple example. To this end, consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}. \quad (92)$$

The eigenvalues of  $\mathbf{A}$  are

$$\lambda_{1,2} = \pm\sqrt{3}i, \quad (93)$$

while the eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 + \sqrt{3}i \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 - \sqrt{3}i \end{bmatrix}. \quad (94)$$

Denote by  $\bar{\lambda}_i, \bar{\mathbf{v}}_i$  the complex conjugates of the eigenvalues and eigenvectors. Clearly, for  $i = 1, 2$

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad \Rightarrow \quad \overline{\mathbf{A}\mathbf{v}_i} = \overline{\lambda_i\mathbf{v}_i} \quad \Rightarrow \quad \mathbf{A}\bar{\mathbf{v}}_i = \bar{\lambda}_i\bar{\mathbf{v}}_i, \quad (95)$$

i.e., if  $\mathbf{v}_i$  is an eigenvector corresponding to  $\lambda_i$  then  $\bar{\mathbf{v}}_i$  is an eigenvector corresponding to  $\bar{\lambda}_i$ . So, in practice, we just need to compute one eigenvector of  $\mathbf{A}$ , since the other one is going to be the complex conjugate of such vector. To compute the *real Jordan form*, we simply replace the complex eigenvectors (94) with the real and imaginary component of one vector<sup>8</sup>, i.e., we consider the real basis

$$\mathbf{P} = \begin{bmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{bmatrix} \quad (97)$$

We have

$$\mathbf{AP} = \underbrace{\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{bmatrix}}_{\mathbf{P}} = \begin{bmatrix} 0 & 2\sqrt{3} \\ -3 & -\sqrt{3} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{bmatrix}}_{\mathbf{J}} \quad (98)$$

Hence the *real Jordan form*<sup>9</sup> is the skew-symmetric matrix

$$\mathbf{J} = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{bmatrix} \quad (101)$$

and the similarity transformation (97) has real entries. Of course, we are also allowed to consider the transformation

$$\mathbf{P} = \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{3} \end{bmatrix}, \quad (102)$$

which yields the real Jordan form

$$\mathbf{J} = \begin{bmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}. \quad (103)$$

If a  $2 \times 2$  matrix  $\mathbf{A}$  has complex conjugate eigenvalues of the form

$$\lambda_{1,2} = \mu \pm i\omega \quad (104)$$

then the real Jordan form of  $\mathbf{A}$  is

$$\mathbf{J} = \begin{bmatrix} \mu & \pm\omega \\ \mp\omega & \mu \end{bmatrix}. \quad (105)$$

<sup>8</sup>Note that the real component of both vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in (94) is

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{while the imaginary component is } \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}. \quad (96)$$

<sup>9</sup>On the other hand, the *complex Jordan form* is obtained by the methods we studied in the course note 4. In fact the matrix  $\mathbf{A}$  is diagonalizable. Hence, we have

$$\mathbf{J} = \begin{bmatrix} \sqrt{3}i & 0 \\ 0 & -\sqrt{3}i \end{bmatrix} \quad (99)$$

and the (complex) similarity transformation

$$\mathbf{P} = \begin{bmatrix} -2 & -2 \\ 1 - \sqrt{3}i & 1 + \sqrt{3}i \end{bmatrix}. \quad (100)$$

## Conservative and gradient systems

**Conservative systems.** We say the  $n$ -dimensional nonlinear dynamical system

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \quad (1)$$

is *conservative* if there exists a scalar field  $E(\mathbf{x})$ , i.e., a function from  $\mathbb{R}^n$  into  $\mathbb{R}$ , that is constant<sup>1</sup> along each trajectory of (1) and non-constant in any open set  $D \subseteq \mathbb{R}^n$ . This definition implies that trajectories of a conservative system (1) are level sets of  $E(\mathbf{x})$ , i.e.,

$$E(\mathbf{X}(t, \mathbf{x}_0)) = E(\mathbf{x}_0), \quad (2)$$

where  $\mathbf{X}(t, \mathbf{x}_0)$  is the trajectory generated by (1) for some given  $\mathbf{x}_0$ .

**A condition for a system to be conservative.** Consider the system (1). How do we check if the system is conservative, i.e., if there exists a scalar field  $E(\mathbf{x})$  that is preserved along trajectories? By definition  $E(\mathbf{x})$  is preserved along trajectories if and only if

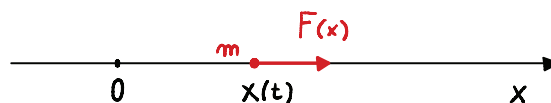
$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \nabla E(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt} = 0 \quad \Rightarrow \quad \nabla E(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = 0. \quad (3)$$

Hence, a system is conservative if and only if we can find a function  $E(\mathbf{x})$  the gradient of which is orthogonal to  $\mathbf{f}(\mathbf{x})$  at each point  $\mathbf{x}$ . In other words, the system (1) is conservative if there exists a solution to the following partial differential equation (PDE)

$$\nabla E(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = 0, \quad (4)$$

where  $E(\mathbf{x})$  is unknown and  $\mathbf{f}(\mathbf{x})$  is the vector field at the right hand side of (1).

**Point mass moving on a line.** Consider a point particle moving on a straight line under the action of an external force  $F(x)$  depending only on the position  $x$  of the particle.



The equation of motion is the Newton's second law

$$m \frac{d^2x}{dt^2} = F(x), \quad (5)$$

where  $m$  is the mass of the particle. We have seen in the course note 1 that  $F(x)$  can be always written in terms of a *potential energy function*  $V(x)$  as

$$F(x) = -\frac{dV(x)}{dx}. \quad (6)$$

---

<sup>1</sup>The constant depend on the particular trajectory, i.e., on the initial condition  $\mathbf{x}_0$  (see Eq. (2)).

Multiplying (5) by the velocity  $dx(t)/dt$  of the particle and collecting terms yields

$$\frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + V(x) \right] = 0. \quad (7)$$

Hence, the quantity

$$\underbrace{E \left( x, \frac{dx}{dt} \right)}_{\text{total energy}} = \underbrace{\frac{1}{2} m \left( \frac{dx}{dt} \right)^2}_{\text{kinetic energy}} + \underbrace{V(x)}_{\text{potential energy}}, \quad (8)$$

is conserved along trajectories of (5). Upon definition of  $x_1(x) = x(t)$  and  $x_2(t) = dx(t)/dt$ , we can re-write (5) as a two-dimensional system of first-order ODEs

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{1}{m} \frac{dV(x_1)}{dx_1} \end{cases} \quad (9)$$

This allows us to write the energy function (8) as

$$E(x_1, x_2) = \frac{1}{2} m x_2^2 + V(x_1). \quad (10)$$

Note that the gradient of energy function  $E(x_1, x_2)$  is orthogonal to the vector field at the right hand side of (9). In fact,

$$\nabla E(x_1, x_2) \cdot \mathbf{f}(\mathbf{x}) = \left( \frac{dV}{dx_1}, m x_2 \right) \cdot \left( x_2, -\frac{1}{m} \frac{dV}{dx_1} \right) = \frac{dV}{dx_1} x_2 - \frac{dV}{dx_1} x_2 = 0. \quad (11)$$

Therefore condition (4) is satisfied.

**Duffing oscillator.** The Duffing equation is a non-linear second-order differential equation that models certain oscillators. In its simplest form the equation can be written as <sup>2</sup>

$$\frac{d^2 x}{dt^2} = x - x^3, \quad (13)$$

The “potential energy” corresponding to  $x - x^3$  is

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \quad (14)$$

We rewrite the system (13) as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_1 - x_1^3 \end{cases} \quad (15)$$

---

<sup>2</sup>The full form of the Duffing equation has a friction term proportional to  $\dot{x}$  and a sinusoidal driving force

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t) \quad (12)$$

This system can exhibit very complex dynamics, including chaotic phase similarities and recurrences (see, e.g., C. Bonatto, J. A. C. Gallas, and Y. Ueda, “Chaotic phase similarities and recurrences in a damped-driven Duffing oscillator” *Phys. Rev. E*, **77**, 026217 (2008)).



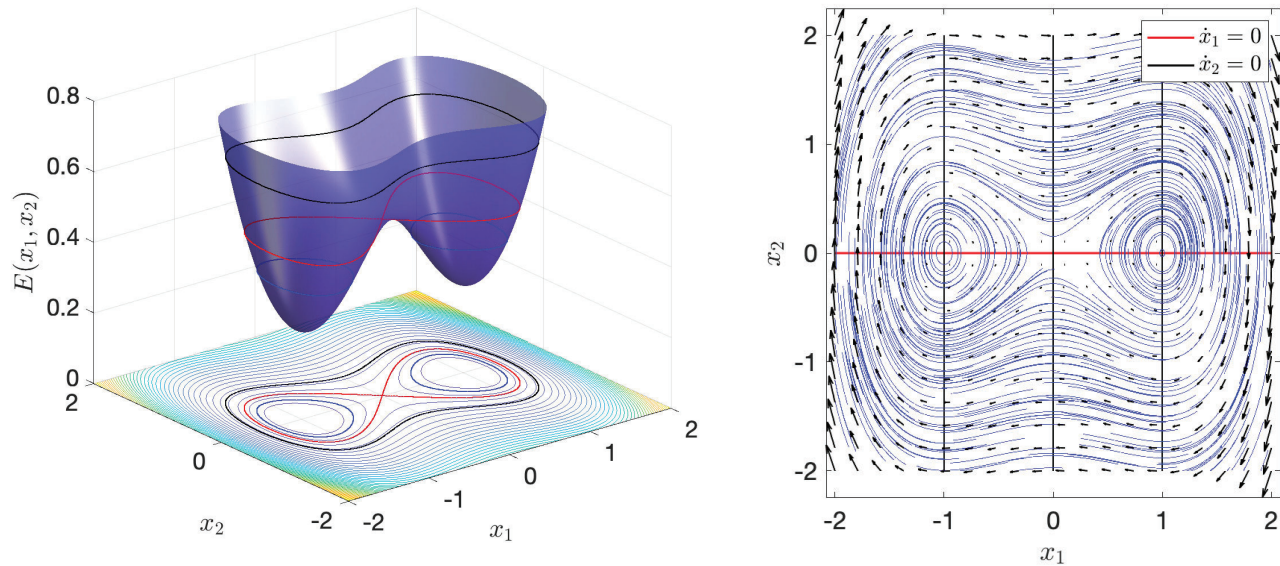


Figure 1: The Duffing oscillator (13) is a conservative system. Shown are the energy function (17) and some of its level sets. Such level sets are trajectories of the system (15). This means that we can construct the phase portrait of the system by taking sections of the energy function and then projecting such level sets onto the  $(x_1, x_2)$  plane. The Duffing oscillator has three fixed points: one saddle node at  $(x_1, x_2) = (0, 0)$  and two centers at  $(x_1, x_2) = (\pm 1, 0)$ .

Multiplying the second equation by  $x_2$  yields

$$x_2 \frac{dx_2}{dt} = (x_1 - x_1^3)x_2 = -\frac{dV(x_1)}{dt} \quad \Rightarrow \quad \frac{d}{dt} \left[ \frac{x_2^2}{2} + V(x_1) \right] = 0 \quad (16)$$

Hence, the following quantity is conserved along the trajectories

$$E(x_1, x_2) = \frac{1}{2} (x_2^2 - x_1^2) + \frac{x_1^4}{4}, \quad (17)$$

i.e., trajectories are level sets of  $E(x_1, x_2)$ . As before the gradient of  $E(x_1, x_2)$  satisfies the orthogonality condition (4). In Figure 1 we demonstrate the correspondence between the levels sets of the energy function (17) and trajectories of the system (15).

**Nonlinear pendulum.** Consider the pendulum sketched in Figure 2. We know from classical mechanics that the equation of motion is (see the course note 3 for a derivation)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin(\theta). \quad (18)$$

As before, set  $x_1(t) = \theta(t)$  and  $x_2(t) = d\theta(t)/dt$ . This allows us to write the second-order nonlinear ODE (18) as a two-dimensional dynamical system

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{g}{L} \sin(x_1) \end{cases} \quad (19)$$

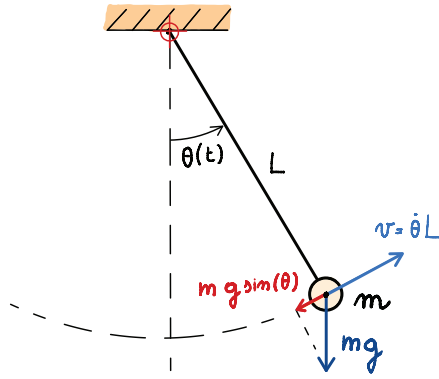


Figure 2: Sketch of a pendulum. The pendulum is assumed to have no friction, i.e., the only external force acting on the point mass  $m$  is gravity.

The “potential energy” corresponding to the force  $F(x) = -\frac{g}{L} \sin(x_1)$  is (modulus an arbitrary constant)

$$V(x_1) = -\frac{g}{L} \cos(x_1). \quad (20)$$

Multiplying the second equation in (19) by  $x_2$  we obtain

$$x_2 \frac{dx_2}{dt} = \frac{g}{L} \frac{d \cos(x_1)}{dt} \quad \Rightarrow \quad \frac{d}{dt} \left[ \frac{x_2^2}{2} - \frac{g}{L} \cos(x_1) \right] = 0 \quad (21)$$

Therefore the following quantity (total energy)

$$E(x_1, x_2) = \frac{1}{2} x_2^2 - \frac{g}{L} \cos(x_1) \quad (22)$$

is preserved along any trajectory of the system (19). This implies that the trajectories of the pendulum in the phase space  $(x_1, x_2)$  (cylindrical phase space) are level sets of (22) (see Figure 3). As before the gradient of  $E(x_1, x_2)$  satisfies the orthogonality condition (4).

**Dynamics of a point mass in three-dimensional space.** Consider Newton’s second law

$$m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}) \quad (23)$$

describing the dynamics of a point mass  $m$  subject to the external force  $\mathbf{F}(\mathbf{x})$ . Such force depends only on the position of the particle, and it could be due to gravity, electromagnetic fields, etc. If the force  $\mathbf{F}(\mathbf{x})$  is conservative (see Appendix A for a review of conservative vector fields), then there exists a potential energy  $V(\mathbf{x})$  such that

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}). \quad (24)$$

This allows us to write equation (23) as

$$m \frac{d^2 \mathbf{x}}{dt^2} = -\nabla V(\mathbf{x}). \quad (25)$$

By taking the scalar (dot) product of the vector equation (23) by  $d\mathbf{x}/dt$  we obtain

$$m \frac{d\mathbf{x}}{dt} \cdot \frac{d^2 \mathbf{x}}{dt^2} = m \left( \frac{dx_1}{dt} \frac{d^2 x_1}{dt^2} + \frac{dx_2}{dt} \frac{d^2 x_2}{dt^2} + \frac{dx_3}{dt} \frac{d^2 x_3}{dt^2} \right) = -\frac{dx_1}{dt} \frac{\partial V}{\partial x_1} - \frac{dx_2}{dt} \frac{\partial V}{\partial x_2} - \frac{dx_3}{dt} \frac{\partial V}{\partial x_3}, \quad (26)$$

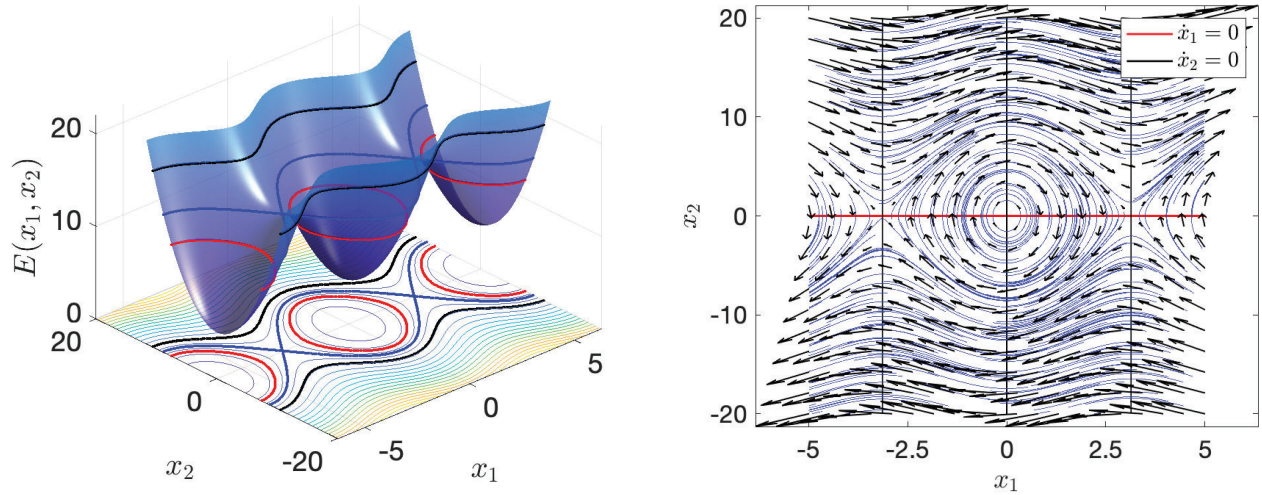


Figure 3: Nonlinear pendulum (18) for  $g = 9.8 \text{ m/s}^2$ , and  $L = 0.5 \text{ m}$ . Shown are the energy function (22) and some of its level sets. Such level sets are trajectories of the system (18). This means that we can construct the phase portrait of the system by taking sections of the energy function and projecting the corresponding level sets onto the  $(x_1, x_2)$  plane. The pendulum has two fixed points repeating periodically (with period  $2\pi$ ): a saddle node at  $(x_1, x_2) = (\pi, 0)$  and a center at  $(x_1, x_2) = (0, 0)$ .

i.e.,

$$\frac{d}{dt} \left( \frac{1}{2} m \|\dot{\mathbf{x}}\|^2 + V(\mathbf{x}) \right) = 0. \quad (27)$$

Once again, the total energy (sum of kinetic and potential energy)

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m \|\dot{\mathbf{x}}\|_2^2 + V(\mathbf{x}) \quad (28)$$

is conserved along trajectories if  $\mathbf{F}(\mathbf{x})$  in (23) is a potential vector field.

**Dynamics of a point mass on a surface.** Consider a point mass moving without friction on a surface  $S(x_1, x_2)$  under the effect of gravity (Figure 4). The coordinates of the point mass in the three dimensional space are

$$(x_1(t), x_2(t), S(x_1(t), x_2(t))). \quad (29)$$

This implies that the velocity  $\mathbf{v}(t)$  of the particle on the surface can be expressed as

$$\mathbf{v}(t) = \left( \dot{x}_1(t), \dot{x}_2(t), \frac{\partial S(x_1(t), x_2(t))}{\partial x_1} \dot{x}_1(t) + \frac{\partial S(x_1(t), x_2(t))}{\partial x_2} \dot{x}_2(t) \right) \quad (30)$$

Hence, the kinetic energy of the particle is

$$\begin{aligned} T(\dot{x}_1, \dot{x}_2, x_1, x_2) = & \frac{1}{2} m \left[ \dot{x}_1^2 + \dot{x}_2^2 + \left( \frac{\partial S(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial S(x_1, x_2)}{\partial x_2} \dot{x}_2 \right)^2 \right] \\ & \frac{1}{2} m \left[ \dot{x}_1^2 \left( 1 + \frac{\partial S(x_1, x_2)}{\partial x_1} \right) + \dot{x}_2^2 \left( 1 + \frac{\partial S(x_1, x_2)}{\partial x_2} \right) + 2\dot{x}_1\dot{x}_2 \frac{\partial S(x_1, x_2)}{\partial x_1} \frac{\partial S(x_1, x_2)}{\partial x_2} \right]. \end{aligned} \quad (31)$$

On the other hand, the potential energy due to gravity is

$$V(x_1, x_2) = mgS(x_1, x_2). \quad (32)$$

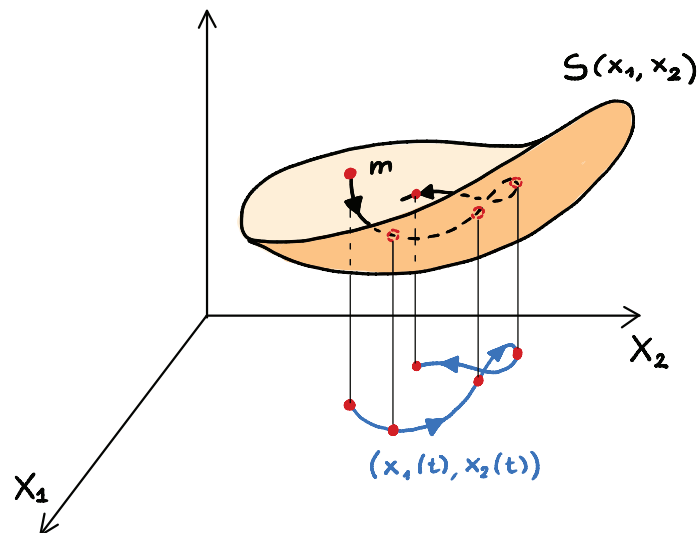


Figure 4: Dynamics of a point mass on a surface  $S(x_1, x_2)$  under the effect of the gravity.

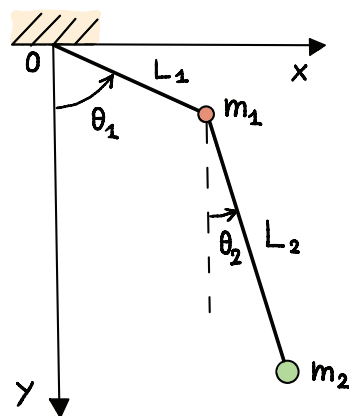


Figure 5: Sketch of a double pendulum. Gravity acts on both masses. There is also an interaction between  $m_1$  and  $m_2$  through the rod of length  $L_2$ .

The total energy of the system is

$$E(\dot{x}_1, \dot{x}_2, x_1, x_2) = T(\dot{x}_1, \dot{x}_2, x_1, x_2) + V(x_1, x_2), \quad (33)$$

where  $T$  and  $V$  are given in (31) and (32). Since there is no friction  $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$  is conserved along trajectories, i.e., the system is conservative. Hence, the trajectories are level sets of  $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$ . In other words, the relation between the coordinates  $(x_1, x_2)$  and velocity  $(\dot{x}_1, \dot{x}_2)$  of the point mass  $m$  is fully determined by the level sets of  $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$ . The value of the level set depends on the initial condition of the particle (position and velocity). In the case where the surface  $S$  is a sphere, this system is called *spherical pendulum*. In the next course note on Lagrangian and Hamiltonian dynamics we will see how to derive the equations of motion for this system.

**Double pendulum.** Consider the double pendulum sketched in Figure 5 We can express the position

of the point masses  $m_1$  and  $m_2$  relative to the coordinate system  $(x, y)$  as

$$\begin{cases} x_1 = L_1 \sin(\theta_1) \\ y_1 = L_1 \cos(\theta_1) \end{cases} \quad \begin{cases} x_2 = x_1 + L_2 \sin(\theta_2) \\ y_2 = y_1 + L_2 \cos(\theta_2) \end{cases} \quad (34)$$

The total potential energy is the sum of the potential gravitational energies of both masses

$$V(\theta_1, \theta_2) = (m_1 + m_2)gL_1(1 - \cos(\theta_1)) + m_2gL_2(1 - \cos(\theta_2)). \quad (35)$$

On the other hand, the kinetic energy is

$$\begin{aligned} T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right). \end{aligned} \quad (36)$$

Note that the kinetic energy is a quadratic form in  $\dot{\theta}_1$  and  $\dot{\theta}_2$ . In this case, the total energy of the system can be written as

$$\begin{aligned} E(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) + V(\theta_1, \theta_2) \\ &= \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) + \\ &\quad (m_1 + m_2)gL_1(1 - \cos(\theta_1)) + m_2gL_2(1 - \cos(\theta_2)). \end{aligned} \quad (37)$$

Remarkably, the trajectories of the double pendulum can be chaotic. Yet, they are level sets of the four-dimensional energy function (37). In the next course note on Lagrangian and Hamiltonian dynamics we will see how to derive the equations of motion for this system.

**Properties of conservative systems.** Trajectories of conservative systems are level sets of some energy function  $E(\mathbf{x})$  (see Equation (2)). Based on this observation it is straightforward to prove the following properties

- Relative maxima and relative minima of the energy function  $E(\mathbf{x})$  are fixed points of the system. Moreover, trajectories nearby such fixed points are necessarily closed. In two dimensions maxima and minima of the energy function  $E(\mathbf{x})$  are *centers* (see Figure 1 and Figure 3). In higher dimensions such local maxima and minima define a closed trajectory on a high-dimensional torus.
- Conservative systems cannot have attractors or repellers. This includes stable/unstable nodes, spirals, limit cycles, and any other attracting or repelling set (e.g., fractal attractors).
- Saddle nodes in 2D conservative system are saddle points of the energy function. This is clearly demonstrated in Figure 1 and Figure 3.

**Gradient systems.** We say that the  $n$ -dimensional dynamical system (1) is a gradient system if it can be written as

$$\frac{d\mathbf{x}}{dt} = -\nabla V(\mathbf{x}). \quad (38)$$

In other words, a gradient system is a first-order dynamical system in which the vector field  $\mathbf{f}(\mathbf{x})$  is conservative<sup>3</sup> (see Appendix A for a review of conservative vector fields). The function  $V(\mathbf{x})$  in (38) is called potential energy of  $\mathbf{f}(\mathbf{x})$ . Note that  $V(\mathbf{x})$  is the opposite of the potential function  $\varphi(\mathbf{x})$  discussed

<sup>3</sup>It is important to distinguish second-order dynamical systems with potential forces (e.g., (25)) from first-order gradient system of the form (38). Gradient systems are, in general, *not* conservative.

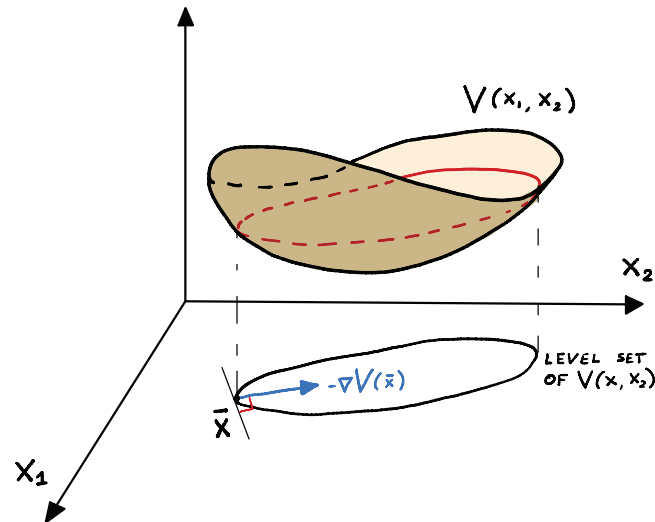


Figure 6: Geometric meaning of the gradient  $-\nabla V(x_1, x_2)$ .

in Appendix A. To check if a dynamical system is a gradient system, we simply need to check if  $\mathbf{f}(\mathbf{x})$  is conservative (conditions given in Appendix A).

As shown in Figure 6, the vector  $-\nabla V(\mathbf{x})$  is orthogonal to the level set of  $V(\mathbf{x})$  passing through  $\mathbf{x}$ , and it points in the direction where  $V(\mathbf{x})$  decreases the most. To prove orthogonality, consider a point  $\mathbf{x}$  and parameterize the level set  $V(\mathbf{x})$  via a curve  $\mathbf{s}(\lambda)$  such that  $\mathbf{s}(0) = \mathbf{x}$  ( $\lambda$  is the parameter of the curve). By expanding  $\mathbf{s}(\lambda)$  in a Taylor series at  $\lambda = 0$  yields

$$\mathbf{s}(\lambda) = \mathbf{s}(0) + \frac{d\mathbf{s}(0)}{d\lambda}\lambda + \dots \quad (39)$$

Evaluate  $V(\mathbf{x})$  along the curve (39) to obtain

$$V(\mathbf{s}(\lambda)) = V(\mathbf{s}(0)) + \nabla V(\mathbf{s}(0)) \cdot \frac{d\mathbf{s}(0)}{d\lambda}\lambda + \dots \quad \Rightarrow \quad \nabla V(\mathbf{x}) \cdot \frac{d\mathbf{s}(0)}{d\lambda} = 0. \quad (40)$$

This implies that  $\nabla V(\mathbf{x}) \perp d\mathbf{s}(0)/d\lambda$ , i.e., that  $\nabla V(\mathbf{x})$  is orthogonal to the level set of  $V(\mathbf{x})$  passing through  $\mathbf{x}$ . To see why, simply recall that the vector  $d\mathbf{s}(0)/d\lambda$  is tangent to the curve  $\mathbf{s}(\lambda)$  at  $\mathbf{s}(0) = \mathbf{x}$ . On the other hand, trajectories of conservative systems are level sets of some energy function.

Gradient systems arise very often in optimization theory, e.g., when computing the (relative) minimum of a function  $V(\mathbf{x})$ . In this context, (38) represents a continuous version of the so-called *gradient descent* scheme, in which the relative minimum of a cost function  $V(\mathbf{x})$  is computed iteratively via

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \gamma \nabla V(\mathbf{x}_k), \quad (41)$$

where  $\gamma > 0$  is a real number called “learning rate”.

Clearly, the critical points of  $V(\mathbf{x})$ , i.e., maxima, minima, saddles, and any other point  $\mathbf{x}^*$  satisfying

$$\nabla V(\mathbf{x}^*) = \mathbf{0}$$

are fixed points of the system (38). Moreover,  $V(\mathbf{x})$  decreases monotonically along trajectories of the system (38) (recall that  $\nabla V$  is orthogonal to the level sets of  $V$ ). This can be also shown by a direct

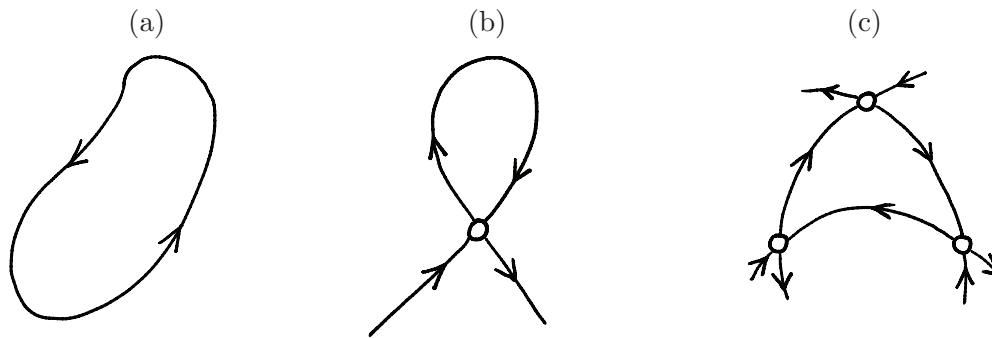


Figure 7: Gradient systems cannot have limit cycles (a), homoclinic orbits (b), or heteroclinic cycles (c).

calculation,

$$\begin{aligned}
 V(\mathbf{x}(t_2)) &= V(\mathbf{x}(t_1)) + \int_{t_1}^{t_2} \frac{dV(\mathbf{x}(t))}{dt} dt \quad t_2 \geq t_1 \\
 &= V(\mathbf{x}(t_1)) + \int_{t_1}^{t_2} \nabla V(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}(s)}{dt} dt \\
 &= V(\mathbf{x}(t_1)) - \int_{t_1}^{t_2} \nabla V(\mathbf{x}(t)) \cdot \nabla V(\mathbf{x}(t)) dt \\
 &= V(\mathbf{x}(t_1)) - \int_{t_1}^{t_2} \|\nabla V(\mathbf{x}(t))\|_2^2 dt \\
 &\leq V(\mathbf{x}(t_1)),
 \end{aligned} \tag{42}$$

where the equality sign holds if and only if  $\|\nabla V(\mathbf{x}(t))\|_2 = 0$ , i.e., only at fixed points of (38). Equation (42) rules out the possibility of any closed orbit in gradient systems, including limit cycles, homoclinic orbits, and heteroclinic cycles (see Figure 7).

The Jacobian matrix of a gradient system is necessarily *symmetric* (see Eq. (44)). Therefore it can only have *real eigenvalues*. This implies that the flow of a gradient system nearby a hyperbolic fixed points cannot have any rotating component due to complex conjugate eigenvalues, i.e., no spirals. In other words, hyperbolic fixed points of gradient systems can only be sources, sinks, or saddle nodes!

## Appendix A: Conservative vector fields

Let  $\mathbf{F}(\mathbf{x})$  be a continuously differentiable vector field defined in a *simply connected* domain<sup>4</sup>  $D \subseteq \mathbb{R}^n$ . We say that  $\mathbf{F}(\mathbf{x})$  is conservative if it can be written as a gradient of some function  $\varphi : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , i.e.,

$$\mathbf{F}(\mathbf{x}) = \nabla \varphi(\mathbf{x}). \tag{43}$$

There are several equivalent conditions that characterize conservative vector fields in a simply connected domain:

- a) The Jacobian of  $\mathbf{F}(\mathbf{x})$  is symmetric, i.e.,

$$\frac{\partial F_i(\mathbf{x})}{\partial x_j} = \frac{\partial F_j(\mathbf{x})}{\partial x_i} \quad \text{for all } \mathbf{x} \in D \text{ and for all } i, j = 1, \dots, n. \tag{44}$$

In two and three dimensions this condition is equivalent to the well known condition expressing the fact that  $\mathbf{F}(\mathbf{x})$  is “irrotational” or curl-free.

<sup>4</sup>A domain  $D \subseteq \mathbb{R}^n$  is called simply connected if any closed curve in  $D$  can be shrunk to a point without getting out of  $D$ .



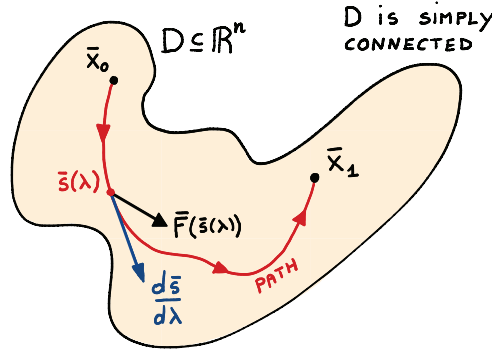


Figure 8: Sketch of a simply connected domain  $D \subseteq \mathbb{R}^n$ . Shown are two points  $\mathbf{x}_0, \mathbf{x}_1 \in D$ , a path  $\mathbf{s}(\lambda)$  connecting  $\mathbf{x}_0 = \mathbf{s}(0)$  to  $\mathbf{x}_1 = \mathbf{s}(1)$  the vector field  $\mathbf{F}(\mathbf{x})$  along the path, and the velocity vector  $d\mathbf{s}(\lambda)/d\lambda$ . If  $\mathbf{F}(\mathbf{x})$  is conservative, then the line integral (45) does not depend on the path connecting  $\mathbf{x}_0$  to  $\mathbf{x}_1$ .

b) The line integral of  $\mathbf{F}(\mathbf{x})$  along a smooth curve  $\mathbf{s}(\lambda)$  in  $D$  connecting two arbitrary points  $\mathbf{x}_0, \mathbf{x}_1 \in D$

$$\varphi(\mathbf{x}_1) = \varphi(\mathbf{x}_0) + \int_0^1 \mathbf{F}(\mathbf{s}(\lambda)) \cdot \frac{d\mathbf{s}(\lambda)}{d\lambda} d\lambda \tag{45}$$

does not depend on the path connecting  $\mathbf{x}_0$  to  $\mathbf{x}_1$  (see Figure 8).

c) The line integral of  $\mathbf{F}(\mathbf{x})$  along any smooth closed curve<sup>5</sup>  $\mathbf{s}(\lambda)$  is zero, i.e.,

$$\int_0^1 \mathbf{F}(\mathbf{s}(\lambda)) \cdot \frac{d\mathbf{s}(\lambda)}{d\lambda} d\lambda = 0 \tag{46}$$

This follows immediately from (46) if we set  $\mathbf{x}_1 = \mathbf{x}_0$ .

It can be shown that (43), (44), and path independence of (45) all imply each other. For example, if the symmetry condition (44) is satisfied for all  $\mathbf{x} \in D$  ( $D$  simply connected) then there exists a potential  $\varphi(\mathbf{x})$  satisfying (43).

The potential  $\varphi(\mathbf{x})$  can be determined by computing the line integral (45) along *any* path connecting two points in  $D$ . In particular, if  $D$  is convex we can consider a simple line as a path

$$\mathbf{s}(\lambda) = (1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}_1 \quad \lambda \in [0, 1]. \tag{47}$$

*Example:* Consider the two dimensional vector field  $\mathbf{F}(\mathbf{x}) = (F_1(x_1, x_2), F_2(x_1, x_2))$  where

$$F_1(x_1, x_2) = 2x_1x_2, \quad F_2(x_1, x_2) = x_1^2. \tag{48}$$

It is straightforward to show that  $\mathbf{F}(\mathbf{x})$  conservative. In fact, the Jacobian of  $\mathbf{F}$  is symmetric for all  $\mathbf{x} \in \mathbb{R}^2$

$$\frac{\partial F_1(x_1, x_2)}{\partial x_2} = 2x_1 = \frac{\partial F_2(x_1, x_2)}{\partial x_1}. \tag{49}$$

Therefore the symmetry condition (44) is satisfied. To compute a potential  $\varphi(\mathbf{x})$  consider a line (47) connecting an arbitrary point  $\mathbf{x}$  to the origin, i.e.,

$$\mathbf{s}(\lambda) = \lambda\mathbf{x} \quad \frac{d\mathbf{s}(\lambda)}{d\lambda} = \mathbf{x}. \tag{50}$$

<sup>5</sup>A closed curve satisfies  $\mathbf{x}_0 = \mathbf{s}(0) = \mathbf{s}(1) = \mathbf{x}_1$ .



We are free to choose any curve we like to evaluate the integral (45) (we already know that the vector field is conservative). We have,

$$\begin{aligned}
 \varphi(\mathbf{x}) &= \varphi(\mathbf{0}) + \int_0^1 \mathbf{F}(\mathbf{s}(\lambda)) \cdot \frac{d\mathbf{s}(\lambda)}{d\lambda} d\lambda \\
 &= \varphi(\mathbf{0}) + \int_0^1 \mathbf{F}(\lambda\mathbf{x}) \cdot \mathbf{x} d\lambda \\
 &= \varphi(\mathbf{0}) + \int_0^1 (2\lambda^2 x_1^2 x_2 + \lambda^2 x_1^2 x_2) d\lambda \\
 &= \varphi(\mathbf{0}) + x_1^2 x_2.
 \end{aligned} \tag{51}$$

The additive constant  $\varphi(\mathbf{0})$  does not change the gradient representing  $\mathbf{F}(\mathbf{x})$  in (43). Therefore we can set it to zero. This yields the potential

$$\varphi(\mathbf{x}) = x_1^2 x_2. \tag{52}$$

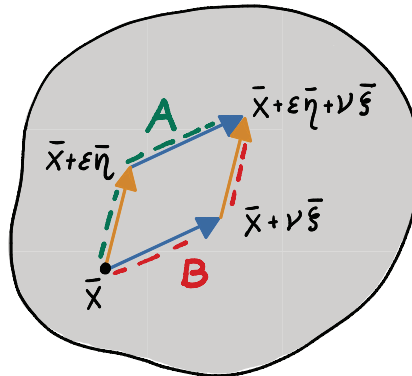
Let us verify that the gradient of (52) coincides with  $\mathbf{F}(\mathbf{x})$ . We have

$$\nabla\varphi(\mathbf{x}) = \left( \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2} \right). \tag{53}$$

yields the vector field (48). We have,

$$\begin{aligned}
 \frac{\partial\varphi}{\partial x_1} &= 2x_1 x_2 && \text{(same as } F_1(x_1, x_2)), \\
 \frac{\partial\varphi}{\partial x_2} &= x_1^2 && \text{(same as } F_2(x_1, x_2)).
 \end{aligned}$$

**Path invariance implies symmetry of the Jacobian.** It is instructive to show how (44) follows directly from (45). To this end, consider a point  $\mathbf{x}$  and its infinitesimal neighborhood. All points in such neighborhood can be connected via two straight lines as follows



By imposing that the line integral of  $\mathbf{F}(\mathbf{x})$  along the paths A and B above are identical we obtain<sup>6</sup>

$$\epsilon \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\eta} + \nu \mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta}) \cdot \boldsymbol{\xi} = \nu \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\xi} + \epsilon \mathbf{F}(\mathbf{x} + \nu\boldsymbol{\xi}) \cdot \boldsymbol{\xi}. \tag{56}$$

<sup>6</sup>Note that along the infinitesimal path (line)  $\mathbf{s}(\epsilon) = \mathbf{x} + \epsilon\boldsymbol{\eta}$

$$\int_0^\epsilon \mathbf{F}(\mathbf{s}(\epsilon)) \cdot \frac{d\mathbf{s}(\epsilon)}{d\epsilon} d\epsilon = \epsilon \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\eta}. \tag{54}$$

Similarly, along the path  $\mathbf{s}(\nu) = \mathbf{x} + \epsilon\boldsymbol{\eta} + \nu\boldsymbol{\xi}$

$$\int_0^\nu \mathbf{F}(\mathbf{s}(\nu)) \cdot \frac{d\mathbf{s}(\nu)}{d\nu} d\nu = \nu \mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta}) \cdot \boldsymbol{\xi}. \tag{55}$$

This equation can be simplified further by expanding  $\mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta})$  and  $\mathbf{F}(\mathbf{x} + \nu\boldsymbol{\xi})$  in a first-order Taylor series, i.e.,

$$\begin{aligned}\mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta}) &\simeq \mathbf{F}(\mathbf{x}) + \epsilon\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\eta}, \\ \mathbf{F}(\mathbf{x} + \nu\boldsymbol{\xi}) &\simeq \mathbf{F}(\mathbf{x}) + \nu\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\xi}.\end{aligned}\tag{57}$$

A substitution of these expansions into (56) yields

$$[\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\eta}] \cdot \boldsymbol{\xi} = [\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\xi}] \cdot \boldsymbol{\eta}.\tag{58}$$

This condition implies (for  $\boldsymbol{\xi}$  and  $\boldsymbol{\eta}$  arbitrary) that the Jacobian of  $\mathbf{F}(\mathbf{x})$  must be a symmetric matrix at  $\mathbf{x}$ .

## Lagrangian and Hamiltonian dynamics

In this course note we provide a brief introduction to Lagrangian and Hamiltonian dynamics, and show some applications. To this end, we consider a system with configuration described by  $n$  *generalized coordinates*

$$\mathbf{q}(t) = (q_1(t), q_2(t), \dots, q_n(t)), \quad (1)$$

and corresponding *generalized velocities*

$$\dot{\mathbf{q}}(t) = (\dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_n(t)). \quad (2)$$

The generalized coordinates  $\mathbf{q}(t)$  can be any set of independent variables, e.g., angles and displacements, that uniquely identify the configuration of the system. The state system is  $2n$ -dimensional and defined by the pair  $(\mathbf{q}, \dot{\mathbf{q}})$

**Kinetic energy, potential energy, and total energy.** We consider dynamical systems in which the kinetic energy can be written as

$$T(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(\mathbf{q}, t) \dot{q}_i \dot{q}_j + \sum_{i=1}^n b_i(\mathbf{q}, t) \dot{q}_i + c(\mathbf{q}, t), \quad (3)$$

where  $a_{ij}(\mathbf{q}, t)$  is a symmetric, positive-definite matrix<sup>1</sup> for all  $\mathbf{q}$  and  $t$ . We will see examples of systems with kinetic in the form (3) later in this note. Furthermore we assume that the potential energy  $V(\mathbf{q}, t)$  depends only on the configuration variables  $\mathbf{q}$  and eventually time. The *total energy* of the system can be written as

$$E(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) + V(\mathbf{q}, t). \quad (4)$$

Note that due to the time dependence, the total energy  $E(\mathbf{q}, \dot{\mathbf{q}}, t)$  may not be conserved in time, i.e., the system may not be conservative. On the other hand, if  $E(\mathbf{q}, \dot{\mathbf{q}}, t)$  does not depend on time then  $E$  is a conserved quantity.

**Lagrangian function and the principle of stationary action.** Define the *Lagrangian function*

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, t), \quad (5)$$

and the *action functional*

$$\mathcal{A}([\mathbf{q}]) = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) dt. \quad (6)$$

The Lagrangian function contains all physical information concerning the system and the forces acting on it. Remarkably, the differential equations governing the dynamics of the system can be derived based on the knowledge of the Lagrangian (5) and the following principle.

- **Principle of stationary action (Hamilton's principle):** The dynamics of a nonlinear dynamical system with Lagrangian function (5) makes the action (6) stationary relative to all possible paths  $\mathbf{q}(t)$  connecting two given admissible configurations  $\mathbf{q}(t_1)$  to  $\mathbf{q}(t_2)$ .

The basic idea of Hamilton's principle is sketched in Figure 1, and it can be expressed mathematically as

$$\delta \mathcal{A}([\mathbf{q}]) = \mathcal{A}([\mathbf{q} + \epsilon \boldsymbol{\eta}]) - \mathcal{A}([\mathbf{q}]) = 0, \quad (7)$$

for arbitrary perturbations  $\boldsymbol{\eta}(t)$  satisfying

$$\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = 0. \quad (8)$$

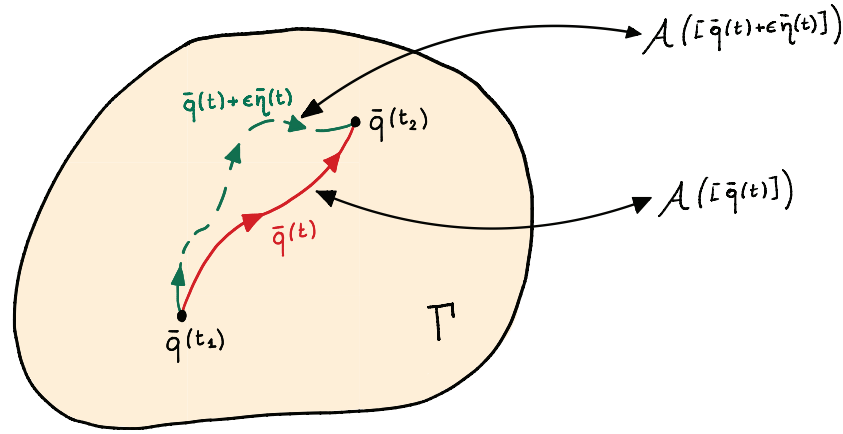


Figure 1: Sketch of the configuration space  $\Gamma$  and two paths connecting two admissible configurations  $\mathbf{q}(t_0)$  and  $\mathbf{q}(t_1)$ . The path taken by the system makes the action (6) stationary (i.e.,  $\delta\mathcal{A}([\mathbf{q}]) = 0$ ) relative to arbitrary small perturbations  $\epsilon\boldsymbol{\eta}(t)$  satisfying  $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}$ .

The quantity  $\delta\mathcal{A}$  is called first variation of the functional  $\mathcal{A}([\mathbf{q}])$  at  $\mathbf{q}(t)$ .

**Euler-Lagrange equations.** Using elementary calculus of variations applied to the action functional (6), it is possible to derive the equations of motion of a nonlinear system just based on the specification of the Lagrangian function (5). This makes the process of deriving such equations very straightforward, and it allows us to proceed in those cases where physical intuition may be difficult to apply. To this end, consider perturbed trajectory

$$\tilde{\mathbf{q}}(t) = \mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t) \quad (9)$$

where  $\boldsymbol{\eta}(t)$  is an arbitrary function satisfying the conditions  $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}$ , so that each path  $\tilde{\mathbf{q}}(t)$  has the same endpoints. By evaluating the action functional (6) at  $\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t)$  we obtain

$$\mathcal{A}([\mathbf{q} + \epsilon\boldsymbol{\eta}]) = \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t), \dot{\mathbf{q}}(t) + \epsilon\dot{\boldsymbol{\eta}}(t), t) dt. \quad (10)$$

The Hamilton's principle can be formulated mathematically as<sup>2</sup>

$$\left. \frac{d\mathcal{A}[\mathbf{q} + \epsilon\boldsymbol{\eta}]}{d\epsilon} \right|_{\epsilon=0} = \frac{d}{d\epsilon} \left[ \int_{t_1}^{t_2} \mathcal{L}(\mathbf{q}(t) + \epsilon\boldsymbol{\eta}(t), \dot{\mathbf{q}}(t) + \epsilon\dot{\boldsymbol{\eta}}(t), t) dt \right]_{\epsilon=0} = 0. \quad (12)$$

Developing the derivatives and evaluating everything at  $\epsilon = 0$  yields

$$\int_{t_1}^{t_2} \sum_{i=1}^n \left[ \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial q_i} \eta_i(t) + \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \dot{q}_i} \dot{\eta}_i(t) \right] dt = 0. \quad (13)$$

By using integration by parts and recalling that the boundary conditions for  $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}$  are zero we can write

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \dot{q}_i} \dot{\eta}_i(t) dt = \underbrace{\left[ \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \dot{q}_i} \eta_i(t) \right]_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \dot{q}_i} \eta_i(t) dt. \quad (14)$$

<sup>1</sup>Recall that positive-definite matrices have strictly positive eigenvalues and therefore they are necessarily invertible.

<sup>2</sup>Note that to first order in  $\epsilon$  we have

$$\delta\mathcal{A}([\mathbf{q}]) = \epsilon \left. \frac{d\mathcal{A}[\mathbf{q} + \epsilon\boldsymbol{\eta}]}{d\epsilon} \right|_{\epsilon=0}. \quad (11)$$

Hence (12) is equivalent to (7).

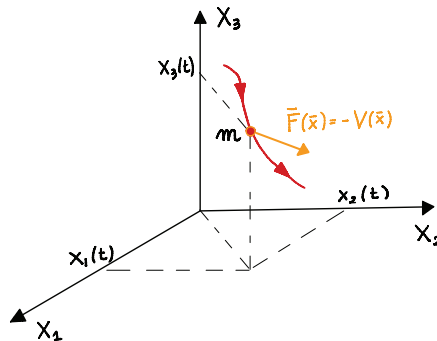


Figure 2: Dynamics of a point mass in 3D under the action of a conservative vector field  $\mathbf{F}(\mathbf{x})$ .

Substituting this back into (13) yields

$$\int_{t_1}^{t_2} \sum_{i=1}^n \left[ \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)}{\partial \dot{q}_i} \right] \eta_i(t) dt = 0. \quad (15)$$

Finally, recalling that  $\eta_i(t)$  is arbitrary we obtain the *Euler-Lagrange equations*

$$\boxed{\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad i = 1, \dots, n.} \quad (16)$$

This is a system of  $n$  second-order nonlinear ODEs. The system is not necessarily in a normal form<sup>3</sup>.

**Dynamics of a point mass in three-dimensional space.** Consider the dynamics of a point mass  $m$  subject to conservative vector field  $\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x})$ . Let

$$\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t)), \quad (18)$$

be the coordinates of the particle relative to a Cartesian coordinate system (see Figure 2). Such coordinates identify the configuration of the system, i.e., they are the variables  $\mathbf{q}(t)$  of this problem. The Lagrangian for this system is

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \underbrace{\frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)}_{\text{kinetic energy}} - V(\mathbf{x}) \quad (19)$$

Differentiating the Lagrangian with respect to  $x_i$  and  $\dot{x}_i$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m \dot{x}_i, \quad \frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial V}{\partial x_i} \quad i = 1, 2, 3 \quad (20)$$

and substituting these expressions into the Euler-Lagrange equations (16) yields

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) = \frac{\partial \mathcal{L}}{\partial x_i} \quad \Rightarrow \quad m \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i} = F_i(\mathbf{x}). \quad (21)$$

Hence, the Hamilton's principle and the corresponding Euler-Lagrange equations for the Lagrangian (19) are completely equivalent to the Newton's equations of motion in this case.

<sup>3</sup>A system of second-order differential equations is said to be in a *normal form* if the second derivatives can be expressed explicitly as

$$\ddot{\mathbf{x}} = \mathbf{g}(\dot{\mathbf{x}}, \mathbf{x}, t). \quad (17)$$

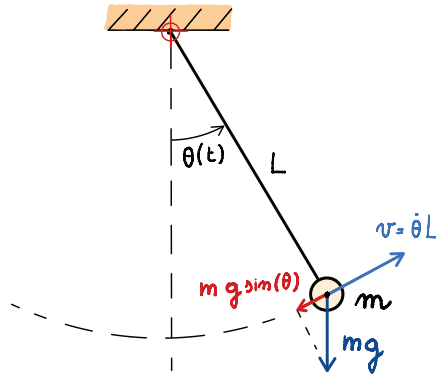


Figure 3: Sketch of a simple pendulum. The pendulum is assumed to have no friction, i.e., the only external force acting on the point mass  $m$  is gravity.

**Nonlinear pendulum.** Consider the pendulum sketched in Figure 3. The configuration variable is  $q(t) = \theta(t)$ , while the state of the system is specified by  $(q, \dot{q}) = (\theta, \dot{\theta})$ . The kinetic energy of the pendulum is

$$T(\dot{\theta}) = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2 \quad (22)$$

while the potential energy relative to the vertical position is

$$V(\theta) = mgL(1 - \cos(\theta)) = -mgL \cos(\theta) + C, \quad (23)$$

where  $C$  is a constant. Hence, the Lagrangian for the pendulum is

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}mL^2\dot{\theta}^2 - mgL(1 - \cos(\theta)). \quad (24)$$

Differentiating the Lagrangian with respect to  $\dot{\theta}$  and  $\theta$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta}, \quad \frac{\partial \mathcal{L}}{\partial \theta} = -mgL \sin(\theta) \quad (25)$$

and substituting the derivatives into the Euler-Lagrange equations (16) yields

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{\partial \mathcal{L}}{\partial \theta} \Rightarrow \ddot{\theta} = -\frac{g}{L} \sin(\theta), \quad (26)$$

i.e., we obtain the differential equation of motion governing the dynamics of the pendulum.

**Double pendulum.** Consider the double pendulum sketched in Figure 4. We choose the configuration variables to be  $(\theta_1(t), \theta_2(t))$ . We have seen in the course note 6 that the potential energy and the kinetic energy for this system can be written as

$$V(\theta_1, \theta_2) = -(m_1 + m_2) \cos(\theta_1) - m_2gL_2 \cos(\theta_2). \quad (27)$$

On the other hand, the kinetic energy is

$$T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2). \quad (28)$$

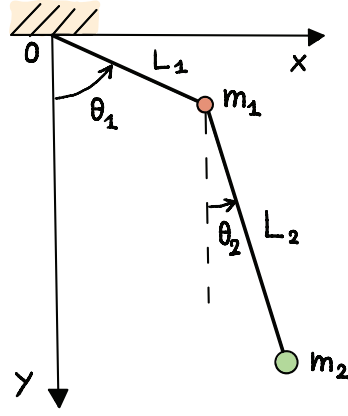


Figure 4: Sketch of a double pendulum. Gravity acts on both masses. There is also an interaction between  $m_1$  and  $m_2$  through the rod of length  $L_2$ .

Hence, the Lagrangian function for this system is

$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2} (m_1 + m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g \cos(\theta_1) + m_2 g L_2 \cos(\theta_2), \quad (29)$$

while the Euler Lagrange equations (16) take the form

$$\begin{cases} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) = \frac{\partial \mathcal{L}}{\partial \theta_1}, \\ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \right) = \frac{\partial \mathcal{L}}{\partial \theta_2}. \end{cases} \quad (30)$$

A rather lengthy calculation of all derivatives in (30) yields the system

$$\ddot{\theta}_1 + a_1(\theta_1, \theta_2) \ddot{\theta}_2 = b_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2), \quad (31)$$

$$\ddot{\theta}_2 + a_2(\theta_1, \theta_2) \ddot{\theta}_1 = b_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2), \quad (32)$$

where

$$a_1(\theta_1, \theta_2) = \frac{L_2}{L_1} \left( \frac{m_2}{m_1 + m_2} \right) \cos(\theta_1 - \theta_2), \quad (33)$$

$$a_2(\theta_1, \theta_2) = \frac{L_1}{L_2} \cos(\theta_1 - \theta_2), \quad (34)$$

and

$$b_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = -\frac{L_2}{L_1} \left( \frac{m_2}{m_1 + m_2} \right) \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{L_1} \sin(\theta_1), \quad (35)$$

$$b_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{L_1}{L_2} \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{g}{L_2} \sin(\theta_2). \quad (36)$$

Note that the system (31)-(32) can be written in a matrix-vector form as

$$\begin{bmatrix} 1 & a_1(\theta_1, \theta_2) \\ a_2(\theta_1, \theta_2) & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} b_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ b_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \end{bmatrix}. \quad (37)$$

Note that the system (37) is not in a normal form, but it can be transformed to a normal form by inverting the matrix multiplying  $(\dot{\theta}_1, \dot{\theta}_2)$  at left hand side of (37), i.e.,

$$\begin{bmatrix} 1 & a_1(\theta_1, \theta_2) \\ a_2(\theta_1, \theta_2) & 1 \end{bmatrix}. \quad (38)$$

Such matrix is invertible since

$$\det \left( \begin{bmatrix} 1 & a_1(\theta_1, \theta_2) \\ a_2(\theta_1, \theta_2) & 1 \end{bmatrix} \right) = 1 - a_1(\theta_1, \theta_2)a_2(\theta_1, \theta_2) = 1 - \left( \frac{m_2}{m_1 + m_2} \right) \cos^2(\theta_1 - \theta_2) > 0. \quad (39)$$

**Hamiltonian dynamics.** Consider the Lagrangian function (5) and define the *canonical momenta*

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad i = 1, \dots, n. \quad (40)$$

A substitution of the Lagrangian (5) into (40) yields

$$p_i = \sum_{j=1}^n a_{ij}(\mathbf{q}, t) \dot{q}_j + b_i(\mathbf{q}, t). \quad (41)$$

The matrix  $a_{ij}(\mathbf{q}, t)$  is, by hypothesis, symmetric and positive definite. Therefore it is always invertible. This implies that the linear system

$$\underbrace{\begin{bmatrix} a_{11}(\mathbf{q}, t) & \cdots & a_{1n}(\mathbf{q}, t) \\ \vdots & \ddots & \vdots \\ a_{n1}(\mathbf{q}, t) & \cdots & a_{nn}(\mathbf{q}, t) \end{bmatrix}}_{\mathbf{A}(\mathbf{q}, t)} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}}_{\dot{\mathbf{q}}} = \underbrace{\begin{bmatrix} p_1 - b_1(\mathbf{q}, t) \\ \vdots \\ p_n - b_n(\mathbf{q}, t) \end{bmatrix}}_{\mathbf{p} - \mathbf{b}(\mathbf{q}, t)} \quad (42)$$

can be uniquely solved for  $\dot{\mathbf{q}}$ . In other words, we can uniquely express the generalized velocities  $\dot{\mathbf{q}}(t)$  as a function of the canonical momenta  $\mathbf{p}(t)$ , and vice versa. This transformation is called *Legendre transformation* and it is written explicitly as

$$\dot{\mathbf{q}} = \mathbf{A}^{-1}(\mathbf{q}, t) [\mathbf{p} - \mathbf{b}(\mathbf{q}, t)]. \quad (43)$$

Next we define the Hamilton's function (or Hamiltonian)

$$\mathcal{H}(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{i=1}^n \underbrace{\frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i}}_{p_i(\mathbf{q}, \dot{\mathbf{q}}, t)} \dot{q}_i - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t). \quad (44)$$

Clearly, by using the Legendre transformation (43) it is possible to write the Hamilton's function as a function of  $\mathbf{p}$ ,  $\mathbf{q}$ , and  $t$ . Hereafter we provide two fundamental theorems of Hamiltonian dynamics.

**Theorem 1** (Hamilton's equations). Consider a nonlinear system with Lagrangian (5) and Hamiltonian (44). The dynamics of the system is governed by the following system of  $2n$  first-order ordinary differential equations in a normal form

$$\begin{cases} \dot{q}_i = \frac{\partial \mathcal{H}(\mathbf{p}, \mathbf{q}, t)}{\partial p_i}, \\ \dot{p}_i = -\frac{\partial \mathcal{H}(\mathbf{p}, \mathbf{q}, t)}{\partial q_i}. \end{cases} \quad (45)$$



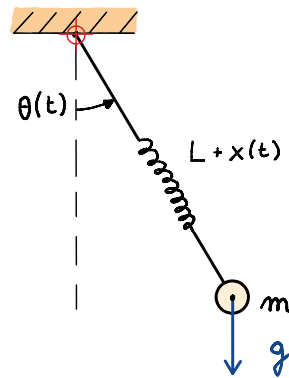


Figure 5: Sketch of an elastic pendulum. The spring can be compressed or extended and it generates a force  $F(x) = -\kappa x$ , where  $\kappa$  is the elastic constant of the spring.

**Theorem 2** (Conservative systems). If the Lagrangian (5) does not explicitly depend on time  $t$ , then the Hamiltonian (44) is conserved along trajectories of the system. In other words, every system described by a time-independent Lagrangian is conservative<sup>4</sup>, and its trajectories are level sets of the Hamiltonian  $\mathcal{H}(\mathbf{p}, \mathbf{q})$ . Note that the Hamiltonian coincides with the total energy of the system.

The proofs of Theorem 1 and Theorem 2 are given in Appendix A. Note that Hamilton's equations are *always* in a normal form. This means that they are always in a form that can be easily integrated numerically.

**Symplectic geometry.** The geometric properties of vector fields associated with Hamiltonian dynamical systems are extremely interesting from a mathematical viewpoint. Indeed, they ended up generating a new research field in mathematics called *symplectic geometry*. Symplectic geometry has its origins in the Hamiltonian formulation of classical mechanics where the phase space of classical systems takes on the structure of a symplectic manifold. It can be shown that Hamiltonian dynamics (both conservative and non-conservative) is always *volume-preserving* in the phase space. This implies that Hamiltonian systems, whether they are conservative or not, cannot have attractors or repellers.

**Hamiltonian formulation of the elastic pendulum.** Consider the elastic pendulum sketched in Figure 5. The configuration space is two-dimensional and defined by the variables  $x(t)$  and  $\theta(t)$ . Hence the phase space is four-dimensional. The kinetic and the potential energy for for this system are, respectively,

$$T = \frac{1}{2}m \left( \dot{x}^2 + (L + x)^2 \dot{\theta}^2 \right), \quad (46)$$

$$V = -mg(L + x) \cos(\theta) + \frac{1}{2}\kappa x^2, \quad (47)$$

where  $\kappa$  is the spring elasticity constant. Hence the Lagrangian  $\mathcal{L} = T - V$  is

$$\mathcal{L}(x, \theta, \dot{x}, \dot{\theta}) = \frac{1}{2}m \left( \dot{x}^2 + (L + x)^2 \dot{\theta}^2 \right) + mg(L + x) \cos(\theta) - \frac{1}{2}\kappa x^2. \quad (48)$$

The canonical momenta (40) are

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \quad (49)$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m(L + x)^2 \dot{\theta}. \quad (50)$$

<sup>4</sup>We proved this result using Nöether's theorem (Eq. 94), as well as a direct calculation (Appendix A).

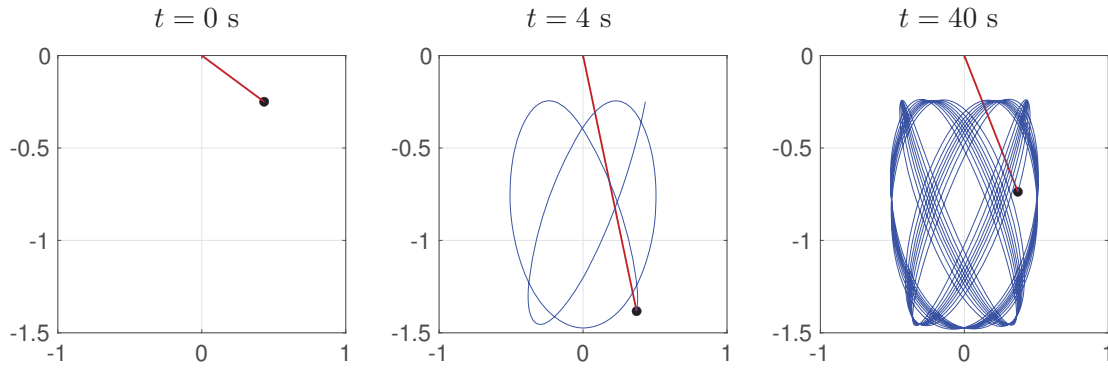


Figure 6: Numerical simulation of the elastic pendulum shown in Figure 5 with mass  $m_1 = 3.5$  kg, length at rest  $L = 0.5$  m, and spring constant  $\kappa = 50$  N/m. The solution is computed by integrating the Hamilton's equations (53) numerically with RK4.

In this case the Legendre transformation (43) is trivial. The Hamiltonian (44) can be written as

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \dot{\theta} - \mathcal{L}(x, \theta, \dot{x}, \dot{\theta}). \quad (51)$$

This can be written in terms of the canonical variables  $p_x$ ,  $p_\theta$ ,  $x$  and  $\theta$  as

$$\begin{aligned} \mathcal{H} &= \frac{p_x^2}{m} + \frac{p_\theta^2}{m(L+x)^2} - \frac{1}{2} \left( \frac{p_x^2}{m^2} + (L+x)^2 \frac{p_\theta^2}{m^2(L+x)^4} \right) - mg(L+x) \cos(\theta) + \frac{1}{2} \kappa x^2 \\ &= \frac{p_x^2}{2m} + \frac{p_\theta^2}{2m(L+x)^2} - mg(L+x) \cos(\theta) + \frac{1}{2} \kappa x^2. \end{aligned} \quad (52)$$

This yields the following system Hamilton's equation

$$\begin{cases} \dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m}, \\ \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{m(L+x)^2}, \\ \dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = \frac{p_\theta^2}{m(L+x)^3} + mg \cos(\theta) - \kappa x, \\ \dot{p}_\theta = -\frac{\partial \mathcal{H}}{\partial \theta} = -mg(L+x) \sin(\theta). \end{cases} \quad (53)$$

The spring-mass pendulum can exhibit chaotic behavior. The nonlinear system of equations (53) is solved numerically in Figure 6. .

**Hamiltonian formulation of the double pendulum.** Consider the double pendulum sketched in Figure 4. The Lagrangian is given in (29) and it is hereafter rewritten for convenience

$$\begin{aligned} \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= \frac{1}{2} (m_1 + m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \\ &\quad (m_1 + m_2) \cos(\theta_1) + m_2 g L_2 \cos(\theta_2). \end{aligned} \quad (54)$$

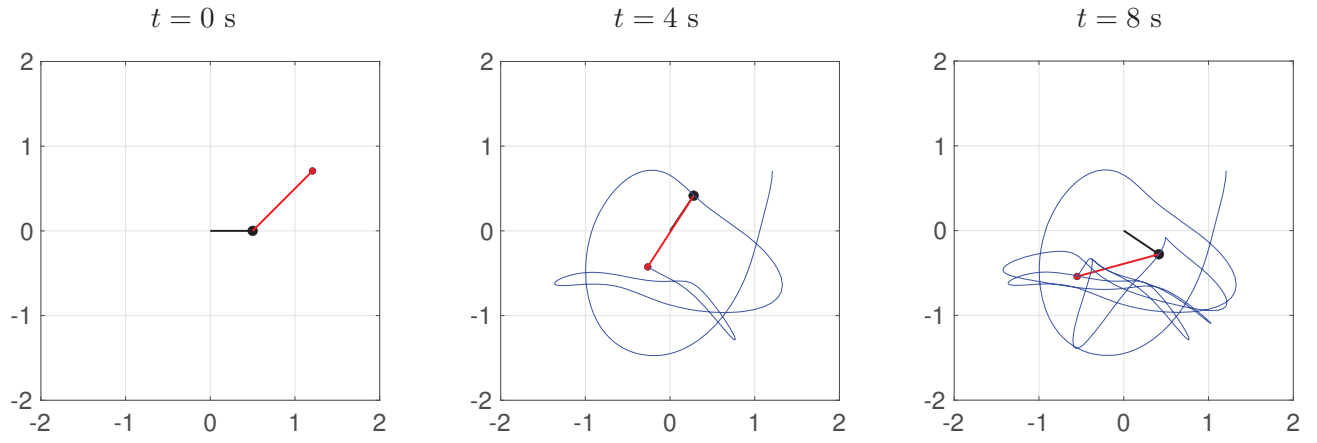


Figure 7: Numerical simulation of a double pendulum with masses  $m_1 = 4$  kg,  $m_2 = 2$  kg and lengths  $L_1 = 1$  m,  $L_2 = 0.5$  m. The solution is computed by integrating the Hamilton's equations (62)-(??) numerically with RK4.

The canonical momenta (40) are

$$p_{\theta_1} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2), \quad (55)$$

$$p_{\theta_2} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2). \quad (56)$$

This linear system can be written as

$$\underbrace{\begin{bmatrix} (m_1 + m_2)L_1^2 & m_2 L_1 L_2 \cos(\theta_2 - \theta_1) \\ m_2 L_1 L_2 \cos(\theta_2 - \theta_1) & m_2 L_2^2 \end{bmatrix}}_{\mathbf{A}(\theta_1, \theta_2)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} p_{\theta_1} \\ p_{\theta_2} \end{bmatrix}. \quad (57)$$

The matrix  $\mathbf{A}(\theta_1, \theta_2)$  is symmetric and positive definite. Therefore it can be inverted to obtain (see Eq. (43))

$$\dot{\theta}_1 = \frac{L_2 p_{\theta_1} - L_1 p_{\theta_2} \cos(\theta_1 - \theta_2)}{L_2 L_1^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))}, \quad (58)$$

$$\dot{\theta}_2 = \frac{-m_2 L_2 p_{\theta_1} \cos(\theta_1 - \theta_2) + (m_1 + m_2) L_1 p_{\theta_2}}{m_2 L_1 L_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))}. \quad (59)$$

The Hamilton's function (44) for this system is

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \dot{\theta}_1 + \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \dot{\theta}_2 - \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2). \quad (60)$$

A substitution of (54)-(56) and (58)-(59) into (60) yields

$$\mathcal{H}(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2) = \frac{m_2 L_2^2 p_{\theta_1}^2 + (m_1 + m_2) L_1^2 p_{\theta_2}^2 - 2m_2 L_1 L_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{2m_2 L_1^2 L_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))} - (m_1 + m_2)g L_1 \cos(\theta_1) - m_2 g L_2 \cos(\theta_2). \quad (61)$$

Hence, the Hamilton's equations of motion are

$$\begin{cases} \dot{\theta}_1 = \frac{\partial \mathcal{H}}{\partial p_{\theta_1}} = \frac{L_2 p_{\theta_1} - L_1 p_{\theta_2} \cos(\theta_1 - \theta_2)}{L_2 L_1^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))}, \\ \dot{\theta}_2 = \frac{\partial \mathcal{H}}{\partial p_{\theta_2}} = \frac{-m_2 L_2 p_{\theta_1} \cos(\theta_1 - \theta_2) + (m_1 + m_2) L_1 p_{\theta_2}}{m_2 L_1 L_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))}, \\ \dot{p}_{\theta_1} = -\frac{\partial \mathcal{H}}{\partial \theta_1} = -(m_1 + m_2) g L_1 \sin(\theta_1) - \alpha(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2), \\ \dot{p}_{\theta_2} = -\frac{\partial \mathcal{H}}{\partial \theta_2} = -m_2 g L_2 \sin(\theta_1) + \alpha(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2), \end{cases} \quad (62)$$

where

$$\begin{aligned} \alpha(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2) &= \frac{p_{\theta_1} p_{\theta_2} \sin(\theta_1 - \theta_2)}{L_1 L_2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))} - \\ &\quad \sin(2(\theta_1 - \theta_2)) \frac{m_2 L_2^2 p_{\theta_1}^2 + (m_1 + m_2) L_1^2 p_{\theta_2}^2 - 2m_1 L_1 L_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{2L_1^2 L_2^2 (m_1 + m_2 \sin^2(\theta_1 - \theta_2))}. \end{aligned} \quad (63)$$

The pendulum system can exhibit chaotic behavior. A numerical solution of Hamilton's equations (62) is shown in Figure 6.

**Two-dimensional Hamiltonian systems.** A two-dimensional Hamiltonian system can be written in the form

$$\begin{cases} \frac{dx_1}{dt} = \frac{\partial \mathcal{H}(x_1, x_2)}{\partial x_2} \\ \frac{dx_2}{dt} = -\frac{\partial \mathcal{H}(x_1, x_2)}{\partial x_1} \end{cases} \quad (64)$$

Hence, given a vector field  $\mathbf{f}(x_1, x_2)$ , if we can find a function  $\mathcal{H}(x_1, x_2)$  such that

$$f_1(x_1, x_2) = \frac{\partial \mathcal{H}(x_1, x_2)}{\partial x_2} \quad f_2(x_1, x_2) = -\frac{\partial \mathcal{H}(x_1, x_2)}{\partial x_1} \quad (65)$$

then we can immediately conclude that the system is conservative. In fact,

$$\begin{aligned} \frac{d\mathcal{H}(x_1(t), x_2(t))}{dt} &= \frac{\partial \mathcal{H}}{\partial x_1} \dot{x}_1 + \frac{\partial \mathcal{H}}{\partial x_2} \dot{x}_2 \\ &= \frac{\partial \mathcal{H}}{\partial x_1} \frac{\partial \mathcal{H}}{\partial x_2} - \frac{\partial \mathcal{H}}{\partial x_2} \frac{\partial \mathcal{H}}{\partial x_1} \\ &= 0. \end{aligned} \quad (66)$$

Note also that the vector field  $\mathbf{f}(x_1, x_2)$  in (65) is divergence-free, i.e.,

$$\nabla \cdot \mathbf{f}(x_1, x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial^2 \mathcal{H}}{\partial x_1 \partial x_2} - \frac{\partial^2 \mathcal{H}}{\partial x_1 \partial x_2}. \quad (67)$$

Therefore the flow generated by (64) preserves volume in phase space. We emphasize that two-dimensional divergence-free vector fields are necessarily Hamiltonian and therefore conservative.

However, conservative systems are not necessarily Hamiltonian, nor volume preserving. It is rather straightforward indeed to manufacture systems that conserve specific quantities but are not Hamiltonian. As an example, consider the function

$$E(x_1, x_2) = x_1^2 x_2 + x_2. \quad (68)$$

A two-dimensional dynamical system conserves  $E(x_1, x_2)$  along a trajectory if and only if

$$\frac{\partial E}{\partial x_1} \dot{x}_1 + \frac{\partial E}{\partial x_2} \dot{x}_2 = 0. \quad (69)$$

Substituting equation (68) into (69) yields

$$2x_1x_2\dot{x}_1 + (x_1^2 + 1)\dot{x}_2 = 0. \quad (70)$$

At this point it is clear that the system

$$\begin{cases} \dot{x}_1 = -x_2^2 \\ \dot{x}_2 = \frac{2x_1x_2^3}{x_1^2 + 1} \end{cases} \quad (71)$$

conserves (68) along any trajectory (and therefore it is a conservative system), but it is not divergence-free nor Hamiltonian. Moreover, it is clear that there exists an infinite number of dynamical systems conserving (68). Another one is

$$\begin{cases} \dot{x}_1 = -\sin(x_2) \\ \dot{x}_2 = \frac{2x_1x_2\sin(x_2)}{x_1^2 + 1} \end{cases} \quad (72)$$

which clearly satisfies condition (70).

## Appendix A: Derivation of the Hamilton's equations

The differential of the Hamilton's function  $\mathcal{H}(\mathbf{p}, \mathbf{q}, t)$  can be written as

$$d\mathcal{H}(\mathbf{p}, \mathbf{q}, t) = \sum_{i=1}^n \left( \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial q_i} dq_i \right) + \frac{\partial \mathcal{H}}{\partial t} dt. \quad (73)$$

On the other hand, by using the definition (44) we obtain

$$d\mathcal{H}(\mathbf{p}, \mathbf{q}, t) = \sum_{i=1}^n \left( dp_i \dot{q}_i + p_i d\dot{q}_i - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt. \quad (74)$$

Using the Euler-Lagrange equations (16) and the definition of canonical momenta (40) we have

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \dot{p}_i. \quad (75)$$

Setting the equality between (73) and (74), and taking into account (75) yields

$$\sum_{i=1}^n \left[ \left( \frac{\partial \mathcal{H}}{\partial p_i} - \dot{q}_i \right) dp_i + \left( \frac{\partial \mathcal{H}}{\partial q_i} + \dot{p}_i \right) dq_i \right] + \left( \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{L}}{\partial t} \right) dt = 0. \quad (76)$$

Assuming that  $dp_i$ ,  $dq_i$  and  $dt$  are arbitrary, we obtain

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}, \quad (77)$$

Which coincide with the Hamilton's equations of motion (45).

Next, we prove that if the Lagrangian function does not depend explicitly on time then the Hamiltonian is conserved along trajectories. To this end, we first notice that if  $\mathcal{L}$  does not explicitly depend on time then  $\mathcal{H}$  does not explicitly depend on time (by the last equation in (77)). Evaluating  $\mathcal{H}(\mathbf{q}, \dot{\mathbf{q}})$  along a trajectory of the system yields

$$\mathcal{H}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = \sum_{i=1}^n \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial \dot{q}_i} \dot{q}_i(t) - \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t)). \quad (78)$$

Differentiating with respect to time yields

$$\frac{d}{dt} \mathcal{H}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = \sum_{i=1}^n \frac{d}{dt} \left( \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial \dot{q}_i} \dot{q}_i(t) \right) - \frac{d}{dt} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t)). \quad (79)$$

Note that

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) &= \sum_{i=1}^n \left( \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial q_i} \dot{q}_i(t) + \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial \dot{q}_i} \ddot{q}_i(t) \right) \\ &= \sum_{i=1}^n \left( \frac{d}{dt} \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial \dot{q}_i} \dot{q}_i(t) + \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial \dot{q}_i} \ddot{q}_i(t) \right) \\ &= \sum_{i=1}^n \frac{d}{dt} \left( \frac{\partial \mathcal{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t))}{\partial \dot{q}_i} \dot{q}_i(t) \right), \end{aligned} \quad (80)$$

where in the second equality we used the Euler-Lagrange equations (16). Substituting (80) into (79) yields

$$\frac{d}{dt} \mathcal{H}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = 0, \quad (81)$$

which proves that the Hamiltonian is conserved along trajectories.

## Nöether's theorem and conservation laws.

Any symmetry of the action functional (6), i.e., an invariance under a transformation group, is associated with a conservation law for the system. This remarkable result was proved by Emmy Nöether in 1918. To formalize Nöether's theorem, let us consider one-parameter group of transformations generated by the dynamical system

$$\begin{cases} \frac{d\mathbf{Q}(\epsilon, \mathbf{q})}{d\epsilon} = \mathbf{G}(\mathbf{Q}(\epsilon, \mathbf{q})) \\ \mathbf{Q}(0, \mathbf{q}) = \mathbf{q} \end{cases} \quad (82)$$

where  $\epsilon$  parameterizes the transformation. The vector field  $\mathbf{G}$  generates the flow  $\mathbf{Q}(\epsilon, \mathbf{q})$ . For small  $\epsilon$  we have the infinitesimal transformation

$$\mathbf{Q}(\epsilon, \mathbf{q}) = \mathbf{q} + \epsilon \mathbf{G}(\mathbf{q}). \quad (83)$$

Setting up the condition for the action functional (6) to be invariant under the transformation group (83)-(91) yields

$$\mathcal{A}([\mathbf{Q}(\epsilon, \mathbf{q})]) = \mathcal{A}([\mathbf{q}]). \quad (84)$$

The first order condition that follows from (84) is

$$\frac{d}{d\epsilon} \mathcal{A}([\mathbf{q} + \epsilon \mathbf{Q}(\mathbf{q})])|_{\epsilon=0} = 0. \quad (85)$$

This can be written as

$$\int_{t_1}^{t_2} \frac{d}{d\epsilon} \mathcal{L} \left( \mathbf{q} + \epsilon \mathbf{G}(\mathbf{q}), \dot{\mathbf{q}} + \epsilon \frac{d\mathbf{G}(\mathbf{q}(t))}{dt}, t \right) \Big|_{\epsilon=0} dt = 0, \quad (86)$$

i.e.,

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \mathbf{G}(\mathbf{q}) + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \frac{d\mathbf{G}(\mathbf{q}(t))}{dt} \right) dt \\ &= \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \cdot \mathbf{G}(\mathbf{q}) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}) \right) dt + \left[ \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}(t)) \right]_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} \underbrace{\left( \frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right) \cdot \mathbf{G}(\mathbf{q})}_{=0 \text{ by Eq. (16)}} dt + \left[ \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}(t)) \right]_{t_1}^{t_2} \\ &= \left[ \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}(t)) \right]_{t_1}^{t_2}. \end{aligned} \quad (87)$$

This implies that the quantity

$$\Lambda = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}) \quad (88)$$

is a constant of motion for the system<sup>5</sup>, i.e., it is a conserved quantity. Such conserved quantities are also called *first integrals* of the system.

<sup>5</sup>Note that (87) implies that

$$\frac{\partial \mathcal{L}(\mathbf{q}(t_2), \dot{\mathbf{q}}(t_2), t_2)}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}(t_2)) = \frac{\partial \mathcal{L}(\mathbf{q}(t_1), \dot{\mathbf{q}}(t_1), t_1)}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}(t_1)) \quad (89)$$

for arbitrary  $t_1$  and  $t_2$ .

More generally, it is possible to consider a transformation group that involves both the configuration variables  $\mathbf{q}$ , i.e., (83) as well as the time variable  $t$  via<sup>6</sup>

$$\tau(\epsilon, t) = t + \epsilon T(t). \quad (91)$$

The perturbed action functional in this case is

$$\mathcal{A}([\mathbf{q} + \mathbf{G}(\mathbf{q})]) = \int_{t_1 + \epsilon T(t_1)}^{t_2 + \epsilon T(t_2)} \mathcal{L} \left( \mathbf{q} + \epsilon \mathbf{G}(\mathbf{q}), \dot{\mathbf{q}} + \epsilon \frac{d\mathbf{G}(\mathbf{q}(t))}{dt}, t + \epsilon T(t) \right) dt. \quad (92)$$

By following the same steps that lead us to (87), i.e., by developing equation (85), it can be shown that invariance of (92) under the transformation (83)-(91) is equivalent to conservation of the following quantity

$$\Lambda = \left( \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - \mathcal{L} \right) T(t) + \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \cdot \mathbf{G}(\mathbf{q}). \quad (93)$$

**Energy conservation.** If  $\mathcal{L}$  does not depend explicitly on time, then of course  $\mathcal{L}$  is invariant under the time translation  $t \rightarrow t + \epsilon$ . Such a time translation is easily generated by setting  $T(t) = 1$  in (91) and (93) (together with  $\mathbf{G}(\mathbf{q}) = \mathbf{0}$ ). Hence, if  $\mathcal{L}$  does not depend explicitly on time then

$$\mathcal{H} = \sum_{k=1}^n \frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_k} \dot{q}_k - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}), \quad (94)$$

is a constant of motion, i.e., it is conserved along trajectories of the system. The function  $\mathcal{H}$  is called *Hamiltonian* of the system.

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<sup>6</sup>The infinitesimal time-warping transformation (91) is defined by the dynamical system

$$\begin{cases} \frac{d\tau(\epsilon, t)}{d\epsilon} = T(\tau(\epsilon, t)) \\ \tau(0, t) = t \end{cases} \quad (90)$$



## Two dimensional nonlinear systems: global theory

We have seen that a great deal of quantitative information about nonlinear dynamical systems can be obtained by “zooming-in” on a small portion of the phase space. Examples of such local analysis are the Hartman-Grobman linearization theorem, the existence of closed trajectories for conservative systems nearby relative maxima and minima of the energy function, etc. We have also discussed results for nonlinear systems that involve a large portion of the phase space. These results go under the umbrella of “global theory” of dynamical systems (see L. Perko, “*Differential equations and dynamical systems*”, Chapter 3), and their proof are often based on topological arguments. Examples of results from global theory are: the center manifold theorem for fixed points<sup>1</sup>, the theorem on the existence and uniqueness of global solutions for dynamical systems, etc.

It is usually harder to obtain information about the dynamics of nonlinear systems on large portions of the phase space. For instance, can you tell if a dynamical system has a limit cycle just by looking at the differential equations? However, for two-dimensional systems of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \quad (1)$$

it is possible to devise theorems that provide global information about the dynamics, i.e., statements about the dynamics within large portions of the phase space. In this course note we discuss some results of such “global theory”, i.e.,

- a) Index theory,
- b) The Poincaré-Bendixson theorem,
- c) The Bendixson criterion to rule out limit cycles,
- d) Liénard systems.

All these results hold for two-dimensional dynamical systems, i.e., there is no Poincaré-Bendixson theorem or index theory for higher dimensional systems.

**Index theory.** Index theory provides global information about phase portraits of two-dimensional dynamical systems. In particular, it allows us to answer questions such as:

- Is there a fixed point inside a closed orbit?
- What types of fixed points can coalesce in a bifurcation?
- Is a certain phase portrait plausible or impossible?

The “index” is a number that is attached to a two-dimensional closed non-intersecting curve (i.e. a Jordan curve) placed in the phase space. Such number represents the number of counterclockwise rotations of the vector field  $\mathbf{f}(\mathbf{x}) = (f_1(x_1, x_2), f_2(x_1, x_2))$  as we proceed counterclockwise along the curve and complete one loop. Let us first define what a Jordan curve is.

**Definition 1** (Jordan curve). A Jordan curve is a closed curve in  $\mathbb{R}^2$  that is homeomorphic to the unit circle. This means that there exists an invertible transformation (continuous with continuous inverse) that maps the unit circle into the Jordan curve and back (see Figure 1).

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<sup>1</sup>There are versions of center manifold theorems for attracting sets other than fixed points, e.g., periodic orbits (see L. Perko, “*Differential equations and dynamical systems*”, Section §3.5).

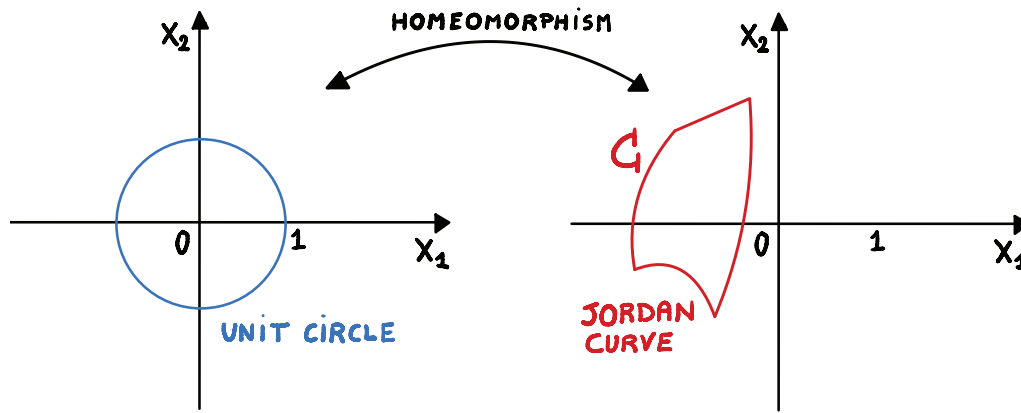


Figure 1: Sketch of a Jordan curve. A Jordan curve cannot have intersections and it is not necessarily differentiable. For instance, it can be just piecewise continuous.

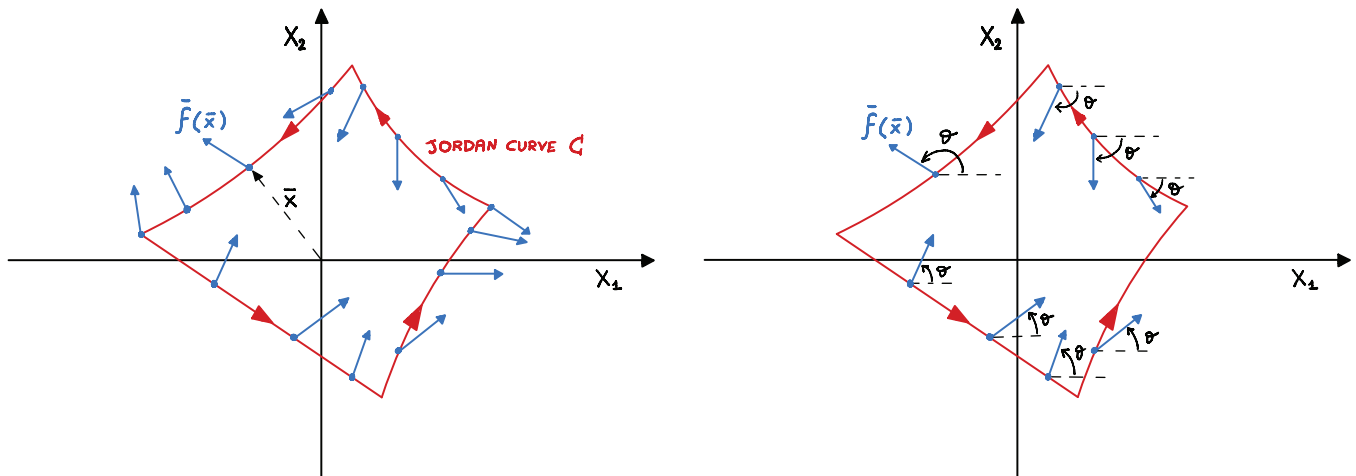


Figure 2: The index of a Jordan curve is the number of counterclockwise rotations of the vector field  $f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))$  in (1) as we loop along the curve counterclockwise and complete one loop.

Based on this definition it is clear that a Jordan curve cannot have intersections with itself, and moreover it is not necessarily differentiable. An example of a Jordan curve is shown in Figure 1.

**Definition 2** (Index of a Jordan curve). Let  $C$  be a Jordan curve. The index of  $C$  is the number of counterclockwise rotations of the vector field  $(f_1(x_1, x_2), f_2(x_1, x_2))$  in (1) as we loop along the curve counterclockwise and complete one round trip.

In Figure 2 we sketch the graphical process of calculating the index of Jordan curve. To calculate the index of  $C$ , we simply integrate the angle  $\vartheta$  (in radians) of  $f(x)$  relative to the horizontal axis as we proceed along the curve  $C$ , i.e.,

$$I_C = \frac{1}{2\pi} \oint d\vartheta. \tag{2}$$

The infinitesimal angle  $d\vartheta$  can be expressed as

$$d\vartheta = \frac{d}{d\lambda} \left[ \arctan \left( \frac{f_2(s_1(\lambda), s_2(\lambda))}{f_1(s_1(\lambda), s_2(\lambda))} \right) \right] d\lambda, \tag{3}$$

where  $s(\lambda) = (s_1(\lambda), s_2(\lambda))$  is a parameterization of the Jordan curve. As an example, the unit circle is a

Jordan curve that can be parameterized as

$$s_1(\lambda) = \cos(2\pi\lambda), \quad s_2(\lambda) = \sin(2\pi\lambda), \quad \lambda \in [0, 1]. \tag{4}$$

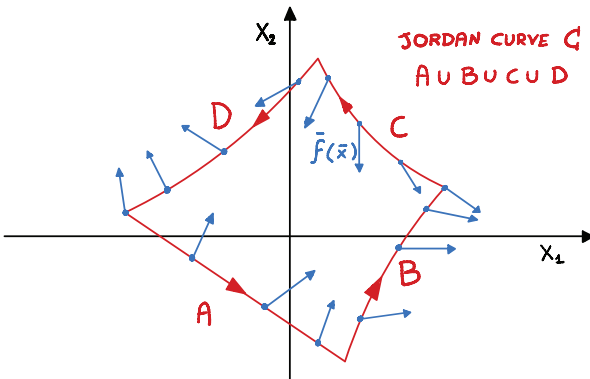
By developing the expression (3) we obtain

$$\begin{aligned} d\vartheta &= \frac{d}{d\lambda} \left[ \arctan \left( \frac{f_2(s_1(\lambda), s_2(\lambda))}{f_1(s_1(\lambda), s_2(\lambda))} \right) \right] d\lambda \\ &= \frac{1}{1 + \left(\frac{f_2}{f_1}\right)^2} \left[ \frac{\partial}{\partial x_1} \left( \frac{f_2}{f_1} \right) \frac{ds_1}{d\lambda} + \frac{\partial}{\partial x_2} \left( \frac{f_2}{f_1} \right) \frac{ds_2}{d\lambda} \right] \\ &= \frac{1}{f_1^2 + f_2^2} \left[ f_1 \left( \frac{\partial f_2}{\partial x_1} \frac{ds_1}{d\lambda} + \frac{\partial f_2}{\partial x_2} \frac{ds_2}{d\lambda} \right) - f_2 \left( \frac{\partial f_1}{\partial x_1} \frac{ds_1}{d\lambda} + \frac{\partial f_1}{\partial x_2} \frac{ds_2}{d\lambda} \right) \right] d\lambda. \end{aligned} \tag{5}$$

Hence, (2) can be written explicitly as

$$I_C = \frac{1}{2\pi} \int_0^1 \frac{1}{f_1^2 + f_2^2} \left[ f_1 \left( \frac{\partial f_2}{\partial x_1} \frac{ds_1}{d\lambda} + \frac{\partial f_2}{\partial x_2} \frac{ds_2}{d\lambda} \right) - f_2 \left( \frac{\partial f_1}{\partial x_1} \frac{ds_1}{d\lambda} + \frac{\partial f_1}{\partial x_2} \frac{ds_2}{d\lambda} \right) \right] d\lambda \tag{6}$$

for any closed curve, i.e., a curve satisfying  $s(0) = s(1)$ . As mentioned above, a Jordan curve can be the union of multiple smooth curves, e.g., four curves  $A$ ,  $B$ ,  $C$ , and  $D$ . In this case,



$$I_C = \frac{1}{2\pi} \left( \int_A d\vartheta + \int_B d\vartheta + \int_C d\vartheta + \int_D d\vartheta \right)$$

*Example:* Consider the two-dimensional vector field  $f(x)$  with components

$$f_1(x_1, x_2) = -x_2, \quad f_2(x_1, x_2) = x_1. \tag{7}$$

Let us compute the index of the curve

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}. \tag{8}$$

A simple parameterization of  $C$  is (see also Eq. (4))

$$s_1(\lambda) = \cos(2\pi\lambda), \quad s_2(\lambda) = \sin(2\pi\lambda), \quad \lambda \in [0, 1]. \tag{9}$$

For this problem, equation (6) takes the form

$$\begin{aligned} I_C &= \frac{1}{2\pi} \int_0^1 \frac{1}{s_1^2(\lambda) + s_2^2(\lambda)} \left[ -s_2(\lambda) \frac{ds_1(\lambda)}{d\lambda} + s_1(\lambda) \frac{ds_2(\lambda)}{d\lambda} \right] d\lambda \\ &= \frac{1}{2\pi} \int_0^1 \frac{2\pi}{\sin^2(2\pi\lambda) + \cos^2(2\pi\lambda)} [\sin^2(2\pi\lambda) + \cos^2(2\pi\lambda)] d\lambda \\ &= 1. \end{aligned} \tag{10}$$

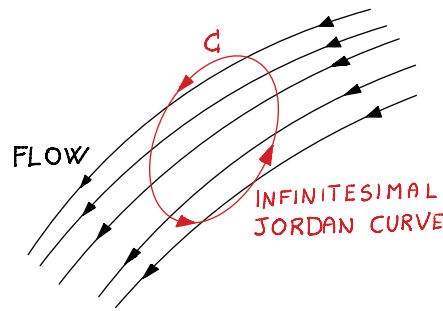


Figure 3: If there is no fixed point inside a Jordan curve  $C$  then the index of  $C$  is zero. In fact, we can continuously shrink  $C$  to an infinitesimal curve that sees the vector field  $\mathbf{f}(\mathbf{x})$  parallel.

Recall that the vector field (7) defines a linear dynamical system with flow corresponding to a center. The trajectories of the system are circles rotating counterclockwise. This explains why the index of the curve  $C$  defined in (8) is equal to one.

There are several theorems characterizing the properties of index of a Jordan curve.

**Theorem 1.** The index of a Jordan curve  $C$  does not change if we continuously deform  $C$  without passing through any fixed points of the dynamical system (1).

The proof of the theorem relies on showing that an arbitrary infinitesimal deformation of the curve  $\mathbf{s}(\lambda)$  leaves  $I_C$  unchanged, provided  $C$  does not intersect with a fixed point). Theorem 1 has several important consequences:

1. The index of a Jordan curve  $C$  does not depend on the curve, but rather it depends only on the fixed points enclosed by the curve.
2. If there is no fixed point inside the curve  $C$  then the index of  $C$  is zero. In fact, we can continuously shrink  $C$  to an infinitesimal circle that sees the vector field  $\mathbf{f}(\mathbf{x})$  parallel (see Figure 3).
3. The index of any closed orbit of a two-dimensional system is one.

Since the index of a Jordan curve depends only on the fixed points enclosed by the curve, it makes sense to calculate the index of all hyperbolic fixed points we know of. To this end we pick an arbitrary Jordan curve enclosing such fixed points and calculate the index. The following theorem summarizes the graphical findings shown in Figure 4.

**Theorem 2** (Index of hyperbolic fixed points). Let  $\mathbf{x}^*$  be a hyperbolic fixed point of the two-dimensional nonlinear system (1). Then for any Jordan curve  $C$  enclosing  $\mathbf{x}^*$

$$I_C = \text{sign}(\det(J_{\mathbf{f}}(\mathbf{x}^*))) = \text{sign}(\lambda_1 \lambda_2), \quad (11)$$

where  $J_{\mathbf{f}}(\mathbf{x}^*)$  is the Jacobian of the system (1) evaluated at  $\mathbf{x}^*$ , and  $\lambda_1, \lambda_2$  are the eigenvalues of  $J_{\mathbf{f}}(\mathbf{x}^*)$ .

For *non-hyperbolic* fixed points the index  $I_C$  can be calculated using equation (2). It is easy to show that if a Jordan curve  $C$  encloses multiple fixed points then its index is the sum of the index.

**Theorem 3.** The index of a Jordan curve enclosing  $n$  fixed points is the sum of the indexes of the fixed points, i.e.,

$$I_C = I_1 + \cdots + I_n. \quad (12)$$

The proof of this theorem is based on the fact that we are free to shrink the black curve  $C$  to the curve shown hereafter (union of blue and red curves) without changing the index. Clearly, the contributions

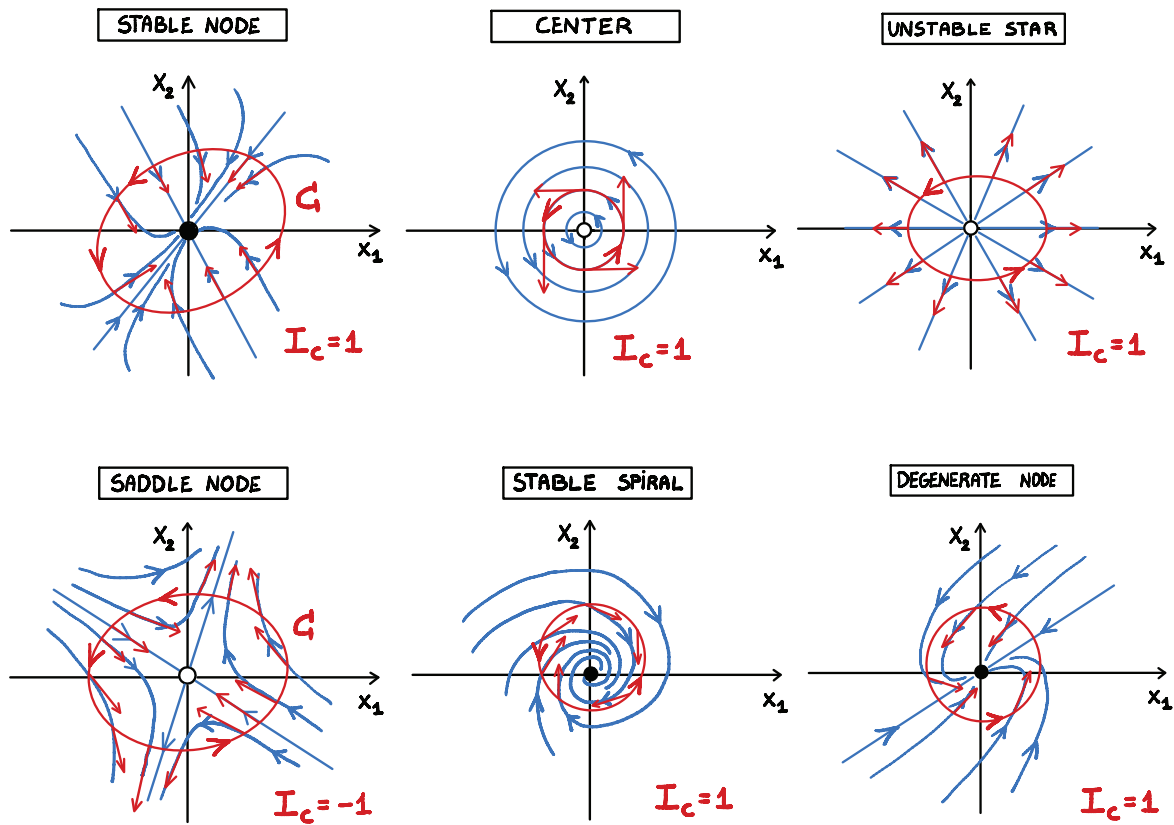
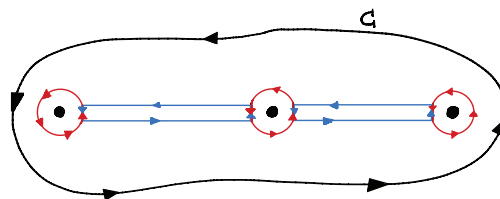


Figure 4: Index  $I_C$  of common fixed points. It is seen that  $I_C = 1$  for spirals, stable and unstable nodes, centers, degenerated nodes and stars.  $I_C = -1$  for saddle nodes.



to the index of the blue curves is zero as such curves do not include any fixed point. Hence, the index of  $C$  is essentially determined by the curves in red, which have index equal to the index of fixed point they enclose.

At this point one can draw all sorts of topological consequences related to the index. For example, a closed orbit of a two-dimensional nonlinear dynamical system must enclose fixed points with indexes that add up to one. In particular, we can have one spiral or one unstable node, but not just one saddle node<sup>2</sup>. It is possible however, that the closed orbit encloses one saddle node and two stable nodes. In fact the index of the sum in this case is  $I_C = 2 - 1 = 1$ , which is compatible with the closed orbit. In Figure 5 we sketch a plausible phase portrait of a closed orbit enclosing one saddle node and two stable nodes.

<sup>2</sup>Recall that the index of a saddle node is  $-1$ .

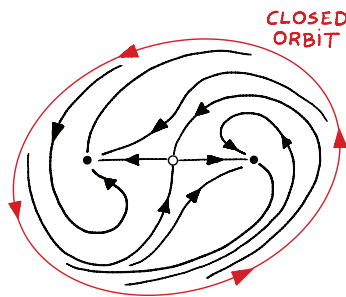


Figure 5: Closed orbit enclosing three fixed points (one saddle node and two stable nodes). The sum of the indexes of all fixed points enclosed by the closed orbit (one saddle node and two stable nodes) is one. It is impossible to have a closed orbit enclosing, e.g., only one saddle node and one stable node as the index of the curve in this case would be zero, contradicting the fact that closed orbits must have index one.

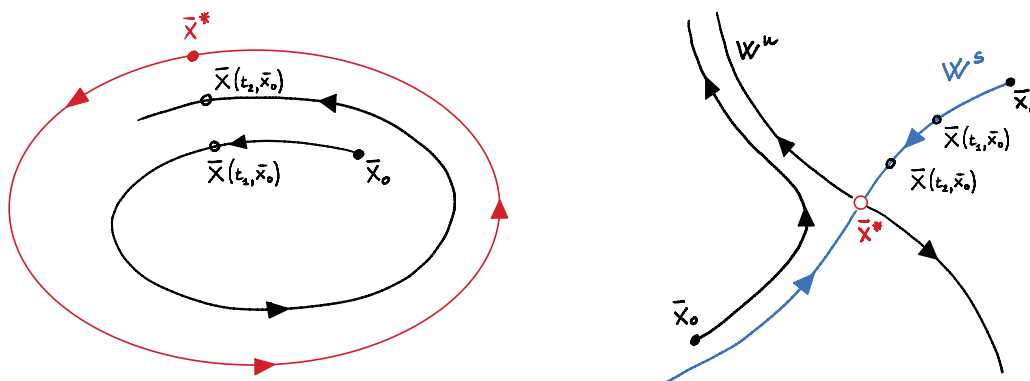


Figure 6: Sketch of two  $\omega$ -limits. The need for a sequence of time instants  $\{t_1, t_2, \dots\}$  in the definition of the  $\omega$ -limit (Definition 3) is clearly explained with reference to the spiraling trajectory.

**Poincaré-Bendixson theorem.** The Poincaré-Bendixson theorem gives us a complete determination of the asymptotic behavior of a large class of flows on the plane, cylinder, two-dimensional sphere, or more generally flows on smooth two-dimensional manifolds. The theorem, which was first conceived by Henri Poincaré in 1892 and finally proved by Ivar Bendixson in 1901, essentially rules out the existence of persistent (i.e., as  $t \rightarrow \infty$ ) aperiodic orbits in bounded regions of any two-dimensional system.

To study the asymptotic behavior of the system (1) it is convenient to introduce the notion of  $\omega$ -limit point of a trajectory and  $\omega$ -limit sets.

**Definition 3** ( $\omega$ -limit set of a trajectory). Let  $\mathbf{X}(t, \mathbf{x}_0)$  be the flow generated by (1). A point  $\mathbf{x}^*$  is called  $\omega$ -limit of a trajectory with initial condition  $\mathbf{x}_0$  if there exists a sequence of time instants<sup>3</sup>  $\{t_1, t_2, \dots\}$  ( $t_i \rightarrow \infty$ ) such that

$$\lim_{i \rightarrow \infty} \mathbf{X}(t_i, \mathbf{x}_0) = \mathbf{x}^*. \tag{13}$$

The set of all  $\omega$ -limit points corresponding to a given  $\mathbf{x}_0$  is called  $\omega$ -limit set of the trajectory with initial condition  $\mathbf{x}_0$  and it denoted as  $\omega(\mathbf{x}_0)$

In Figure 6 we sketch a few  $\omega$  limit sets corresponding to two-dimensional trajectories. More complex limit sets characterized by a finite number of fixed points connected by homoclinic or heteroclinic trajectories are shown in Figure 7.

<sup>3</sup>Note that the sequence of time instants  $\{t_1, t_2, \dots\}$  mentioned in the Definition 3 is not unique.

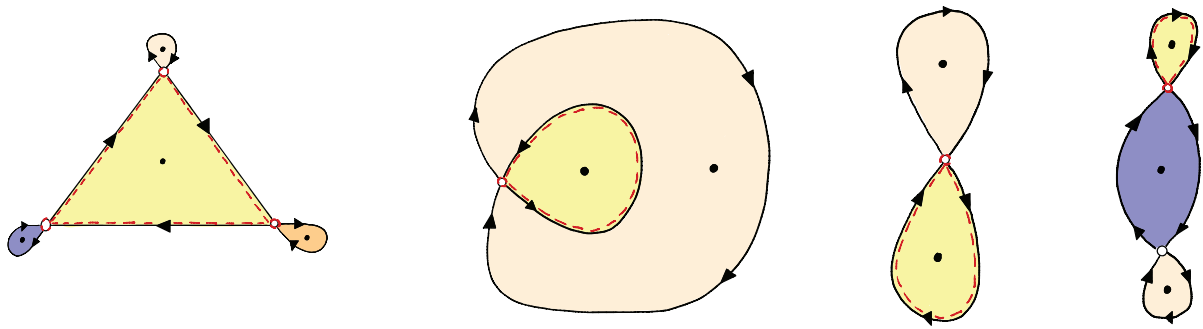


Figure 7: Sketch of  $\omega$ -limit sets characterized by a finite number of fixed points connected by trajectories. The red dashed lines are  $\omega$ -limit sets corresponding to initial conditions in the yellow domains, assuming that the fixed points represented by black dots are repellers.

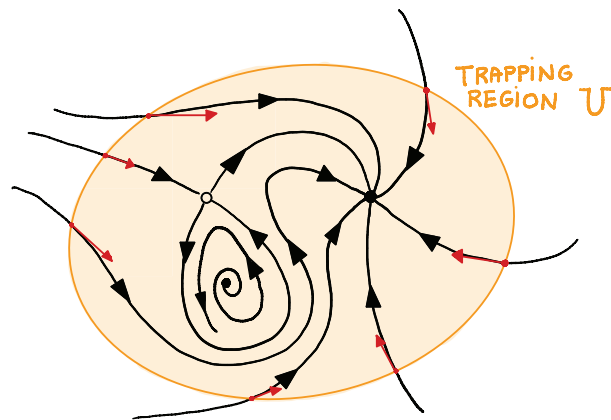


Figure 8: Sketch of a trapping region.

**Definition 4** (Trapping region). Let  $\mathbf{X}(t, \mathbf{x}_0)$  be the flow generated by (1). A compact set  $U \subseteq \mathbb{R}^2$  is called trapping region (or positively invariant set) if

$$\mathbf{X}(t, U) \subseteq U \quad \text{for all } t \geq 0. \tag{14}$$

In other words, a trapping region  $U \subseteq \mathbb{R}^2$  is a region from from which trajectories corresponding to  $\mathbf{x}_0 \in U$  cannot escape  $U$ . This means that the vector field  $\mathbf{f}(\mathbf{x})$  at the boundary of  $U$  is either tangent to the boundary of  $U$ , or it points inward.

If the boundary of the region  $U$  is given as a level set of some function  $F(x_1, x_2)$ , i.e.,  $F(x_1, x_2) = c$  then we can find a simple analytical condition for  $U$  to be a trapping region. In fact, we know that the gradient of  $F$  is orthogonal to the level sets of  $F$ . Moreover, if  $F$  is smaller inside  $U$  and larger outside  $U$  we have that  $\nabla F$  points outward. In this case  $U \subset \mathbb{R}^2$  is a trapping region if

$$\mathbf{f}(\mathbf{x}) \cdot \nabla F(\mathbf{x}) \leq 0, \tag{15}$$

for all points  $\mathbf{x}$  on the level set of  $F(\mathbf{x})$ , i.e., for all points at the boundary of  $U$ .

*Example:* Let us show that the unit disk

$$U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \tag{16}$$

with boundary

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\} \quad (17)$$

is a trapping region for the system

$$\begin{cases} \dot{x}_1 = x_2^2 - x_1 \\ \dot{x}_2 = -x_1x_2 - x_2 \end{cases} \quad (18)$$

To this end, let us first write the circle  $C$  as the zero level set of the function

$$F(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

Note that  $F(0, 0) = -1$  and  $F(2, 0) = 3$ . Therefore  $F$  increases as we go from the interior of  $U$  towards its boundary  $C$  and beyond. The gradient  $\nabla F = (2x_1, 2x_2)$  is orthogonal to  $C$  and in this case it points outward. Let us verify that the condition (15) is satisfied for the region (16) and the vector field (18). We have,

$$\mathbf{f}(\mathbf{x}) \cdot \nabla F(\mathbf{x}) = \dot{x}_1 \frac{\partial F}{\partial x_1} + \dot{x}_2 \frac{\partial F}{\partial x_2} = 2x_1(x_2^2 - x_1) + 2x_2(-x_1x_2 - x_2) = -2(x_1^2 + x_2^2) < 0. \quad (19)$$

Hence  $U$  is a trapping region for the system (18). Note also that the system has a stable node at  $(0, 0)$ .

The next question is, what kind of dynamics can we expect in the trapping region? For instance, can a trajectory in a trapping region wander around it forever without repeating itself (persistent aperiodic behavior)? This question is answered by the Poincaré-Bendixson theorem.

**Theorem 4** (Poincaré-Bendixson). Let  $U \subseteq \mathbb{R}^2$  be a trapping region for the system (1) containing a finite number of fixed points  $\{\mathbf{x}_1^*, \dots, \mathbf{x}_n^*\}$ . Let  $\mathbf{x}_0 \in U$ , and  $\omega(\mathbf{x}_0)$  the  $\omega$ -limit set of the trajectory with initial condition  $\mathbf{x}_0$ . Then one of the following possibilities holds:

1.  $\omega(\mathbf{x}_0)$  is a fixed point;
2.  $\omega(\mathbf{x}_0)$  is a closed orbit;
3.  $\omega(\mathbf{x}_0)$  consists of a finite number of fixed points connected by orbits (see Figure 7).

Of course, if the trapping region has no fixed points in it then the only  $\omega$ -limit set of a point  $\mathbf{x}_0 \in U$  is an attracting closed orbit. Note that it is possible to have multiple closed orbits in a trapping region with no fixed points in it. In fact, Theorem 4 is not saying that the closed orbit is unique, but rather that a trajectory with initial condition  $\mathbf{x}_0$  converges to a closed orbit if there are no fixed points in the trapping region. The closed orbit depends on  $\mathbf{x}_0 \in U$  though. The next question is how many limit cycles<sup>4</sup> can you have in a bounded region of  $\mathbb{R}^2$ ? This is answered by the following theorem due to the French mathematician Henri Dulac (1923).

**Theorem 5** (Dulac's theorem). In any bounded region of the plane, a planar system (1) with  $\mathbf{f}(\mathbf{x})$  analytic in  $\mathbb{R}^2$  has at most a finite number of limit cycles. Any polynomial system has at most a finite number of limit cycles in  $\mathbb{R}^2$ .

**Bendixson's criterion to rule out closed orbits.** By using Gauss's divergence theorem in two-dimensions it is possible to develop a simple criterion to rule out periodic orbits in systems of the form (1). This end, consider a simply connected domain  $D \subset \mathbb{R}^2$ . Let  $\gamma$  be a closed orbit in  $D$ . Such closed orbit identifies a subdomain of  $D$  which is denoted by  $R$  in Figure 9. Since the vector field  $\mathbf{f}(\mathbf{x})$  is tangent to  $\gamma$

<sup>4</sup>A limit cycle is an isolated closed trajectory in phase space. A center is not a limit cycle.



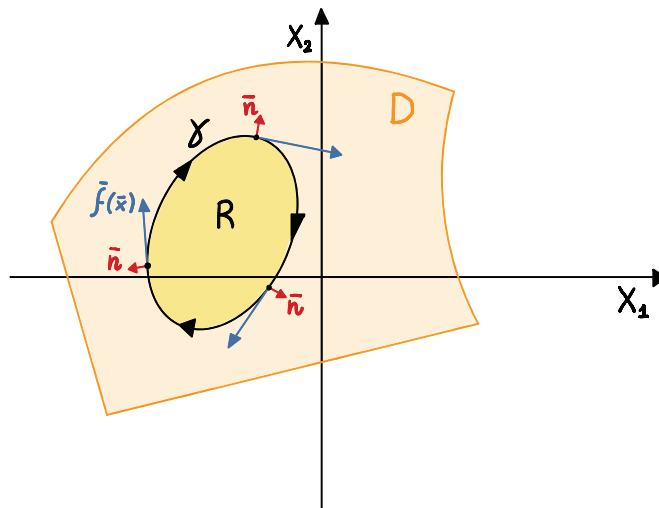


Figure 9: Illustration of the Bendixson's criterion to rule out periodic orbits in two-dimensional system. If the divergence of  $\mathbf{f}(\mathbf{x})$  is not identically zero and does not change sign in the domain  $D$  then (1) has no closed orbits lying entirely in  $D$ . In this sketch the closed orbit is denoted by  $\gamma$ . The vector field  $\mathbf{f}(\mathbf{x})$  is tangent to each point of  $\gamma$ .

at each point we have that the flux  $\mathbf{f}(\mathbf{x})$  through  $\gamma$  (boundary of  $R$ ) is zero. By using Gauss's divergence theorem we obtain

$$\int_R \nabla \cdot \mathbf{f}(\mathbf{x}) d\mathbf{x} = \underbrace{\int_0^1 \mathbf{f}(\mathbf{s}(\lambda)) \cdot \mathbf{n}(\mathbf{s}(\lambda)) d\lambda}_{=0} = 0, \quad (20)$$

where  $\mathbf{s}(\lambda)$  is a parameterization of the closed curve  $\gamma$ , and  $\mathbf{n}$  is the outward unit vector orthogonal to  $\gamma$ . Equation (20) implies the following theorem.

**Theorem 6** (Bendixson's criterion). If on a simply connected region  $D \subseteq \mathbb{R}^2$  the divergence of the vector field  $\mathbf{f}(\mathbf{x})$  is not identically zero and does not change sign, then (1) has no closed orbits lying entirely in  $D$ .

*Example:* The divergence of the system (18) is

$$\nabla \cdot \mathbf{f}(\mathbf{x}) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -x_1 - 2. \quad (21)$$

Hence, by using Bendixson's criterion, we conclude that the system (18) cannot have periodic orbits lying entirely in any simply connected region  $D$  located in the half plane  $x_1 \leq -2$ , or the half plane  $x_1 \geq -2$ .

**Liénard systems.** The Poincaré-Bendixson Theorem can be used to establish the existence of limit cycles for certain two-dimensional systems, provided we can identify a trapping region with no fixed points in it. It is a far more delicate question to determine the exact number of limit cycles of a certain system or class of systems depending on parameters. In this section we discuss a classical result on the uniqueness of a stable limit cycle for systems of the form

$$\ddot{x} + h(x)\dot{x} + g(x) = 0. \quad (22)$$

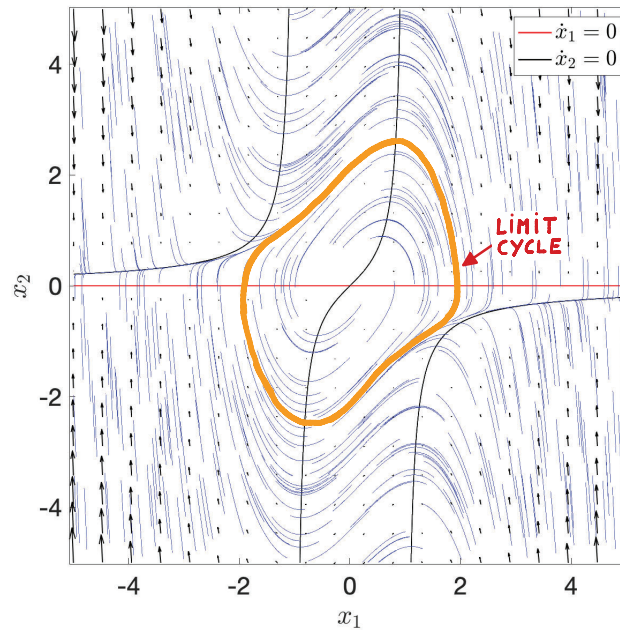


Figure 10: Phase portrait of the Van der Pol oscillator (24) for  $\mu = 1$ .

This result was first established by the French physicist Alfred-Marie Liénard in 1928. To formulate Liénard's theorem, let us first define

$$H(x) = \int_0^x h(z) dz. \quad (23)$$

**Theorem 7** (Liénard's theorem). The system (22) has exactly one stable limit cycle surrounding the origin of the system  $(x, \dot{x}) = (0, 0)$  if the following conditions are satisfied:

1.  $h, g \in C^1(\mathbb{R})$ ;
2.  $h$  is an even function  $h(-x) = h(x)$ , and  $g$  is an odd function  $g(-x) = -g(x)$ ;
3.  $g(x) > 0$  for all  $x > 0$ ;
4. The function  $H(x)$  in (23) has the following properties:
  - There unique zero  $a > 0$  such that  $H(a) = 0$
  - $H(x) < 0$  for all  $0 < x < a$ ,
  - $H$  increases monotonically to infinity for  $x \geq a$ .

*Example (Van der Pol oscillator):* The Van der Pol oscillator was originally proposed by the Dutch electrical engineer and physicist Van der Pol while he was working at Philips. The differential equation defining the oscillator is

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \quad \mu > 0. \quad (24)$$

This is clearly a Liénard system. In fact the functions

$$h(x) = \mu(x^2 - 1) \quad \text{and} \quad g(x) = x, \quad (25)$$

satisfy all conditions listed in Theorem 7. Intuitively, the nonlinear term  $\mu(x^2 - 1)\dot{x}$  is responsible for the creation of the limit cycle. In fact,

$$\begin{aligned} |x| < 1 &\Rightarrow \mu(x^2 - 1) < 0 \Rightarrow \text{negative damping} \Rightarrow \text{small oscillations amplified,} \\ |x| > 1 &\Rightarrow \mu(x^2 - 1) > 0 \Rightarrow \text{positive damping} \Rightarrow \text{large oscillations decay.} \end{aligned}$$

We can rewrite (24) as a two-dimensional system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\mu(x_1^2 - 1)x_2 - x_1 \end{cases} \quad (26)$$

In Figure 10 we plot the phase portrait generated by (26). As easily seen, the fixed point  $(0,0)$  is an unstable spiral. Looking at the orbits of the system, we see that it is not straightforward to identify a trapping region in this case.