Fourier spectral methods

In this course note we discuss Fourier spectral methods for initial/boundary value problems defined in spatial domains with periodic boundary conditions. This subject is discussed extensively, e.g., in [1, §2 and §3].

Trigonometric approximation theory

Let us begin by studying the properties of continuous and discrete Fourier series expansions. For the sake of simplicity, we consider functions of only one variable $u(x)$ defined on $x \in [0, 2\pi]$. The classical Fourier series of a periodic function $u \in L^2([0, 2\pi])$ is given as

$$u(x) = a_0 + \sum_{k=1}^{\infty} a_k \sin(kx) + \sum_{k=1}^{\infty} b_k \cos(kx). \tag{1}$$

By using the Euler's formulas

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}, \qquad \cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}, \tag{2}$$

we can write (1) as[1]

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad c_n \in \mathbb{C}, \tag{4}$$

where expansion coefficients are given by

$$c_p = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ipx} dx, \qquad p \in \mathbb{Z}. \tag{5}$$

The decay rate of c_p depends exclusively on the regularity of $u(x)$ in $[0, 2\pi]$. From the point of view of approximation theory, it is important to understand how well a truncated series expansion of the form

$$u_N(x) = \sum_{k=-N}^{N} c_k e^{ikx} \tag{6}$$

approximates the function $u(x)$. The truncated Fourier series (6) can be seen as the projection of (4) onto the $2N + 1$ dimensional space

$$B_N = \text{span}\left\{ e^{ikx} : |k| \leq N \right\}. \tag{7}$$

By taking the L^2 and L^∞ norm of the difference between (4) and (6) we obtain

$$\|u - u_N\|^2_{L^2([0,2\pi])} = \sum_{|n|>N} |c_n|^2, \tag{8}$$

$$\|u - u_N\|_{L^\infty([0,2\pi])} \leq \sum_{|n|>N} |c_n|. \tag{9}$$

Therefore,

$$\sum_{|n|\leq\infty} |c_n|^2 < \infty \quad \Rightarrow \quad \|u - u_N\|^2_{L^2([0,2\pi])} \to 0 \qquad \text{as } n \to \infty, \tag{10}$$

[1] If the function u is periodic in $[0, L]$ then the Fourier series can be written as

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}. \tag{3}$$

and

$$\sum_{|n|\leq\infty} |c_n| < \infty \quad \Rightarrow \quad \|u - u_N\|_{L^\infty([0,2\pi])} \to 0 \qquad \text{as } n \to \infty. \tag{11}$$

This can be phrased as follows: if $u \in L^2$ (or L^∞) then the Fourier series (6) converges in L^2 (or L^∞) as $N \to \infty$. The decay rate of the errors (8) and (9) depends on the decay rate of the Fourier coefficients c_k. The following theorem characterizes such decay rate.

Theorem 1 (Decay rate of Fourier coefficients). If $u(x)$ and its $(s-1)$ derivatives are continuous in $[0, 2\pi]$, and the s derivative is in $L^2([0, 2\pi])$ then

$$|u_n| \sim n^{-s}. \tag{12}$$

Theorem 1 implies that the rate of decay of the errors (8) and (9) depends on the smoothness of the function $u(x)$. This property is known as *spectral convergence*. If $u(x)$ is infinitely smooth with all derivatives periodic then the Fourier coefficients decay faster than any algebraic function of n. Hereafter we provide rigorous convergence results for the Fourier series (6).

Theorem 2 (Spectral convergence of Fourier series in L^2). For any periodic function[2] $u(x) \in H^s([0, 2\pi])$ there exists a positive constant C independent of N such that

$$\|u(x) - u_N(x)\|_{L^2([0,2\pi])} \leq CN^{-s} \left\| \frac{d^s u}{dx^s} \right\|_{L^2([0,2\pi]}. \tag{14}$$

Theorem 3 (Spectral convergence of Fourier series in L^∞). Let $u(x)$ be a periodic function in $C^s([0, 2\pi])$ (space of continuously differentiable functions up to degree s). Then there exists a positive constant C independent of N such that

$$\|u(x) - u_N(x)\|_{L^\infty([0,2\pi])} \leq CN^{-s+1/2} \left\| \frac{d^s u}{dx^s} \right\|_{L^2([0,2\pi]}. \tag{15}$$

Remark: If $u(x)$ is analytic then the upper bounds in (14) and (15) go to zero exponentially fast with N, i.e., the convergence rate of the Fourier series is *exponential* (see [1, p.36]).

Discrete trigonometric expansion

To construct the Fourier series (6) we need to evaluate $2N + 1$ projection coefficients

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx} dx. \tag{16}$$

In general, these integrals cannot be computed analytically. However, one can resorts to numerical quadrature to evaluate such integrals. Quadrature formulas differ based on the position of the grid points, and the choice of an even or odd number of points.

[2]In Theorem 2, $H^s([0, 2\pi])$ denotes the periodic Sobolev space of degree s, i.e., a function space of periodic functions with norm

$$\|g\|^2_{H^s([0,2\pi])} = \int_0^{2\pi} \left[g(x)^2 + \sum_{q=1}^{s} \left(\frac{d^q g(x)}{dx^q} \right) \right] dx. \tag{13}$$

The even expansion. Define the following evenly-spaced grid with an *even* number N of points

$$x_j = \frac{2\pi j}{N} \qquad j = 0, \dots, N-1. \tag{17}$$

Approximating the integral (16) with the trapezoidal rule yields the following expression for the Fourier coefficient c_n

$$c_n \simeq \frac{1}{N}\sum_{k=0}^{N-1} u(x_j) e^{-inx_j}. \tag{18}$$

How good is this approximation? It turns out that the trapezoidal rule on the grid (17) allows us to integrate exactly any trigonometric polynomial e^{inx} for $|n| < N$, i.e., the quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi} e^{inx} dx = \frac{1}{N}\sum_{j=0}^{N-1} e^{inx_j} \qquad \text{is exact} \quad \text{for all } |n| < N, \tag{19}$$

The proof of (19) is given in [1, Theorem 2.5]. Note that (19) implies that by using the trapezoidal rule with N points we can integrate exactly a Fourier series with $2N-1$ modes, i.e., a Fourier series of the form

$$u_{N-1}(x) = \sum_{|k|\leq N-1} \widetilde{c}_k e^{ikx}. \tag{20}$$

Next, consider the Fourier series

$$u_{N/2}(x) = \sum_{|n|\leq N/2} \widetilde{c}_n e^{inx} \qquad \widetilde{c}_n = \frac{1}{N d_n}\sum_{j=0}^{N-1} u(x_j) e^{-inx_j} \tag{21}$$

where

$$d_n = \begin{cases} 2 & |n| = N/2 \\ 1 & |n| < N/2 \end{cases} \tag{22}$$

Note that[3]

$$\widetilde{c}_{N/2} = \widetilde{c}_{-N/2}. \tag{24}$$

which yields exactly N degrees of freedom for the Fourier modes $\widetilde{c}_n$ (as many as the number of grid points), and justifies the scaling factor $d_{N/2} = d_{-N/2} = 2$.

The expression (21) suggests the following representation of the Fourier series

$$\boxed{u_{N/2}(x) = \sum_{j=0}^{N-1} u(x_j) g_j(x)} \qquad \text{where} \qquad g_j(x) = \sum_{|n|\leq N/2} \frac{1}{N d_n} e^{in(x-x_j)}. \tag{25}$$

It can be shown that g_j can be written as

$$\boxed{g_j(x) = \frac{1}{N}\frac{\sin\left(N\dfrac{x-x_j}{2}\right)}{\tan\left(\dfrac{x-x_j}{2}\right)}.} \tag{26}$$

[3] To prove (24), substitute $n = N/2$ in (21) to obtain

$$\widetilde{c}_{N/2} = \frac{1}{2N}\sum_{j=0}^{N-1} u(x_j) e^{-i\pi j} \qquad \widetilde{c}_{-N/2} = \frac{1}{2N}\sum_{j=0}^{N-1} u(x_j) e^{i\pi j}. \tag{23}$$

Clearly $e^{i\pi j} = e^{-i\pi j}$ for all j, and therefore $\widetilde{c}_{N/2} = \widetilde{c}_{-N/2}$.

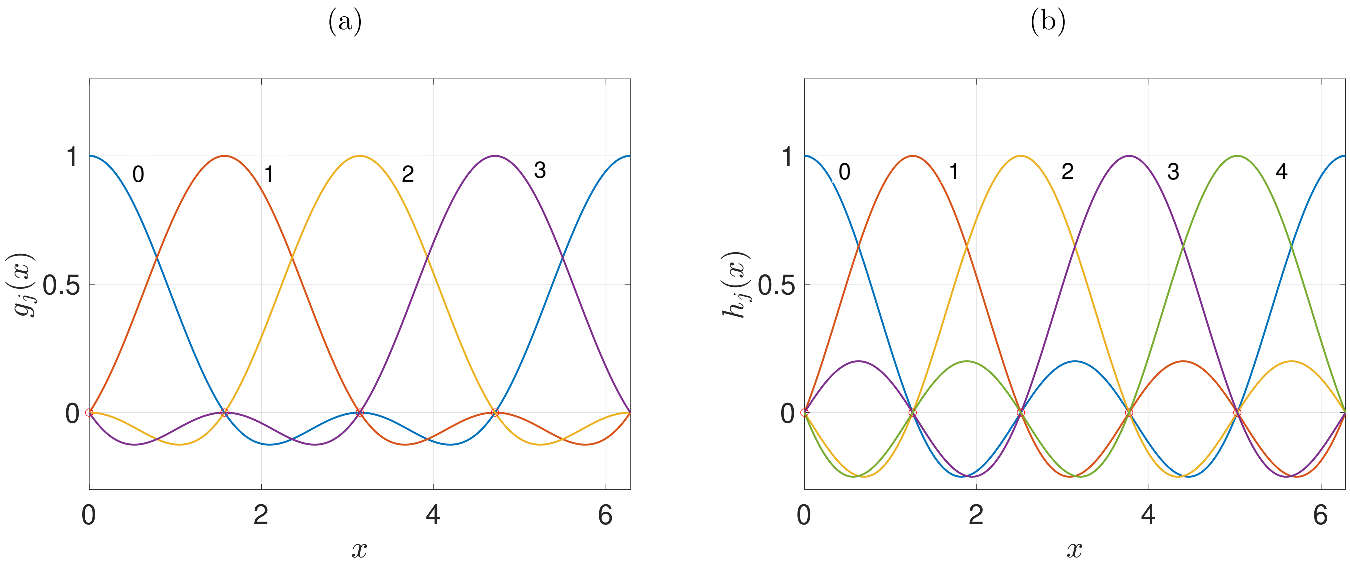

Figure 1: Basis functions for the trigonometric interpolant corresponding to the even expansion (a) (Eq. (26)), and the odd expansion (b) (Eq. (36)). In both cases we set $N = 4$.

It is easy to show that $g_j(x)$ is a cardinal basis for the space of trigonometric functions with frequencies between $-N/2$ and $N/2$. In other words, we have[4]

$$g_j(x_k) = \delta_{kj}. \tag{28}$$

In Figure 1(a) we show the trigonometric basis $\{g_0, \ldots, g_{N-1}(x)\}$ for $N = 4$.

The discrete Fourier series (25) has the form of a Lagrangian interpolant, and it as convergence properties very similar to those of the continuous Fourier series approximation. In particular,the discrete approximation is pointwise convergent for $C^1([0, 2\pi])$ functions and is convergent in L^2 provided only that $u(x) \in L^2([0, 2\pi])$. For example, we have the following convergence result.

Theorem 4 (Spectral convergence of discrete Fourier series in L^2). For any periodic function $u(x) \in H^s([0, 2\pi])$ there exists a positive constant C independent of N such that

$$\|u(x) - u_N(x)\|_{L^2([0,2\pi])} \leq C N^{-s} \left\| \frac{d^s u}{dx^s} \right\|_{L^2([0,2\pi]}. \tag{29}$$

where $u_N(x)$ is defined in (25).

This theorem confirms that the approximation errors of the continuous expansion and the discrete expansion are of the same order.

The discrete expansion (25) allows us to easily compute derivatives at the grid points (17) as

$$\frac{du_{N/2}(x_p)}{dx} = \sum_{j=0}^{N-1} u(x_j) \underbrace{\frac{dg_j(x_p)}{dx}}_{D_{pj}} \tag{30}$$

[4]To prove (28), we recall that

$$\frac{N(x_k - x_j)}{2} = \pi(k - j). \tag{27}$$

which makes the numerator of (28) equal to zero for $k \neq j$. Regarding the case $k = j$ expand both the numerator and denumerator in a Taylor series around $x = x_j$.

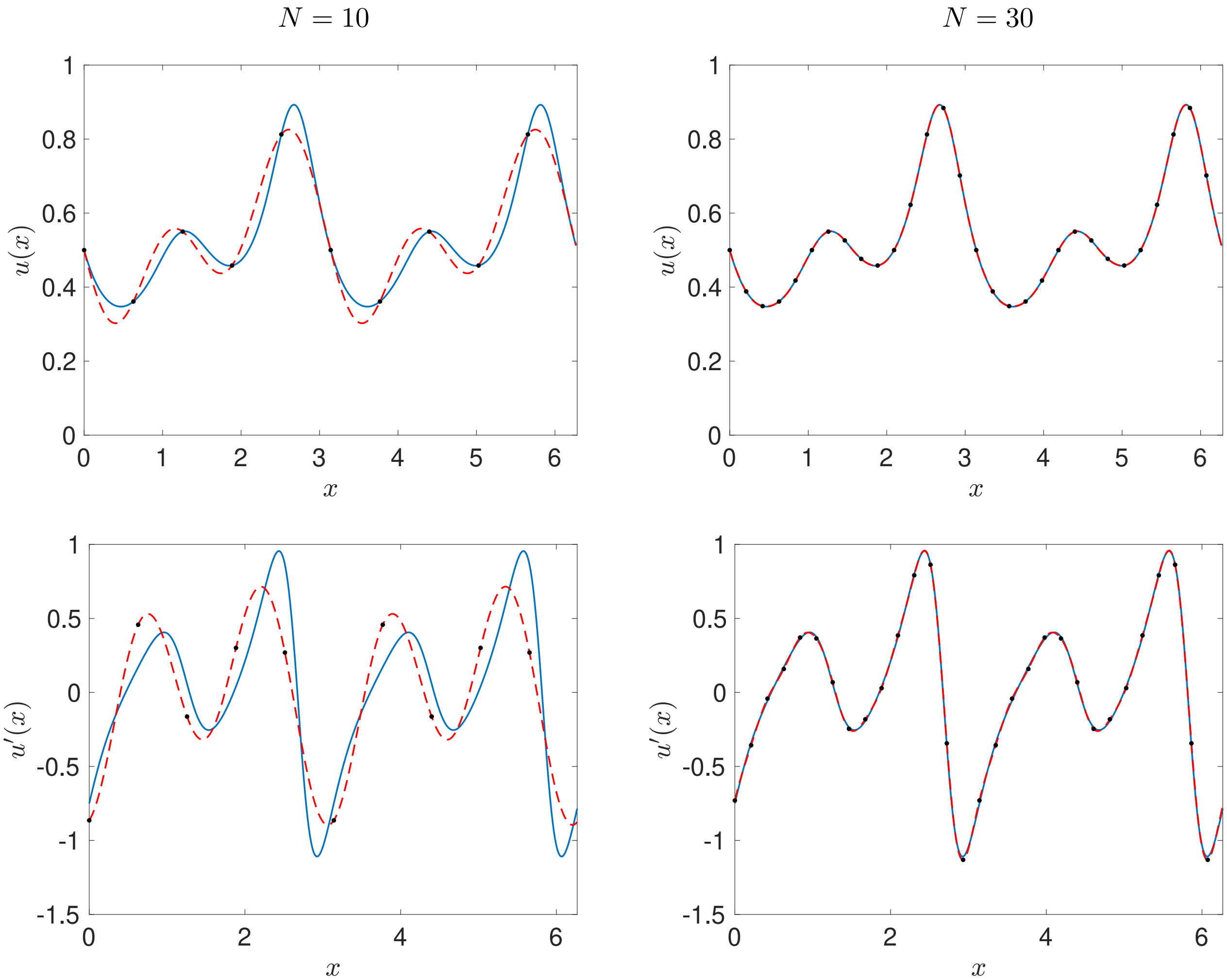

Figure 2: Discrete trigonometric approximation of the function $u(x) = [2 + \cos(x)\sin(3x)]^{-1}$ and its derivative using the even expansion (25)-(26) with $N = 10$ (left column), and $N = 30$ (right column).

where D_{pj} is the Fourier differentiation matrix with entries

$$D_{pj} = \begin{cases} \dfrac{(-1)^{p+j}}{2\tan\left(\dfrac{x_p - x_j}{2}\right)} & p \neq j \\ 0 & p = j \end{cases} \tag{31}$$

In Figure 2 we show the discrete trigonometric approximation of the function $u(x) = [2 + \cos(x)\sin(3x)]^{-1}$ and its derivative using the even expansion (25)-(26) with $N = 10$ (left column), and $N = 30$. In Figure 3 we demonstrate numerically that the L^∞ error of the discrete Fourier series decays exponentially fast with the number of points N. The approximation of higher derivatives follows the exact same route as taken for the first-order derivative. The entries of the second order differentiation matrix $\boldsymbol{D}^{(2)}$, based on an even number of grid points, are

$$D^{(2)}_{pj} = \frac{d^2 g_j(x_p)}{dx^2} = \begin{cases} \dfrac{(-1)^{p+j}}{2}\left[\sin\left(\dfrac{x_p - x_j}{2}\right)\right]^{-2} & p \neq j \\ -\frac{N^2+2}{12} & p = j \end{cases} \tag{32}$$

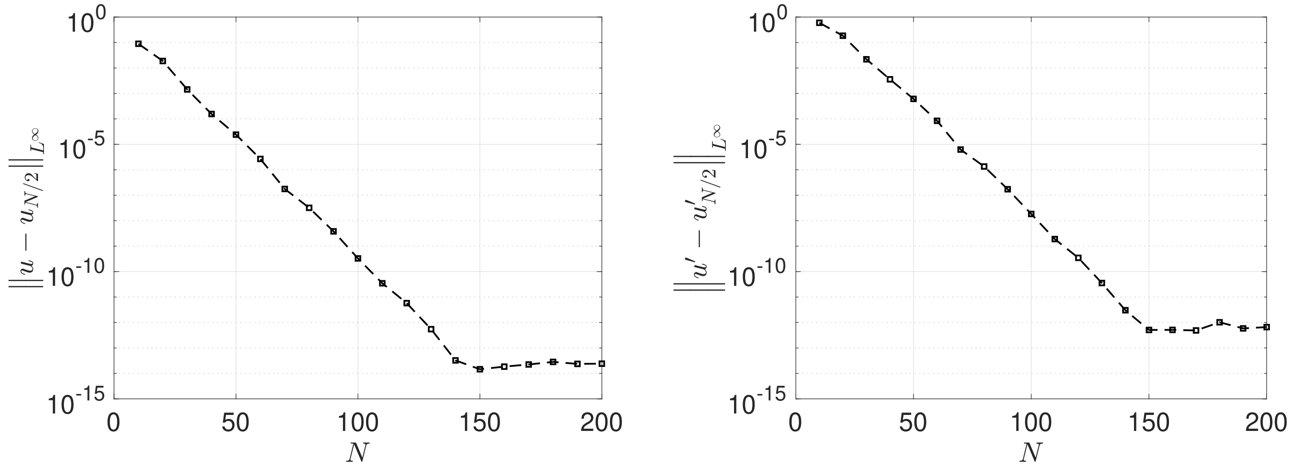

Figure 3: L^∞ error of the discrete trigonometric approximation of the function $u(x) = [2 + \cos(x)\sin(3x)]^{-1}$ and its derivative using the even expansion (25)-(26) versus the number of evenly-spaced points (17).

The odd expansion. Consider an evenly spaced grid in $[0, 2\pi)$ with an odd number of points

$$x_j = \frac{2\pi}{N+1} j \qquad j = 0, \ldots, N \tag{33}$$

where N is even. Using the trapezoidal rule we obtain the following expression for the Fourier coefficients c_n in (16)

$$c_n = \frac{1}{N+1} \sum_{j=0}^{N} u(x_j) e^{-inx_j}. \tag{34}$$

A substitution of c_n into the Fourier series yields the following expression

$$u_{N/2}(x) = \sum_{|n| \leq N/2} c_n e^{inx} = \sum_{j=0}^{N} u(x_j) \underbrace{\sum_{|n| \leq N/2} \frac{e^{in(x-x_j)}}{N+1}}_{h_j(x)}. \tag{35}$$

This can be written as a trigonometric interpolant as

$$\boxed{u_{N/2}(x) = \sum_{j=0}^{N} u(x_j) h_j(x), \qquad \text{where} \qquad h_j(x) = \frac{1}{N+1} \frac{\sin\left(\dfrac{N+1}{2}(x - x_j)\right)}{\sin\left(\dfrac{x - x_j}{2}\right)}.} \tag{36}$$

In Figure 1(b) we show the trigonometric basis $\{h_0, \ldots, h_N(x)\}$ for $N = 4$. To prove the identity

$$h_j(x) = \frac{1}{N+1} \sum_{|n| \leq N/2} e^{in(x-x_j)} = \frac{1}{N+1} \frac{\sin\left(\dfrac{N+1}{2}(x - x_j)\right)}{\sin\left(\dfrac{x - x_j}{2}\right)} \tag{37}$$

let us first recall the definition of Dirichlet kernel[5]

$$D_N(x) = \sum_{n=-N}^{N} e^{ikx} = 1 + 2\sum_{k=1}^{N} \cos(kx) = \frac{\sin\left[\left(N+\dfrac{1}{2}\right)x\right]}{\sin\left(\dfrac{x}{2}\right)}. \tag{40}$$

Clearly,

$$h_j(x) = \frac{1}{N+1}\sum_{|n|\leq N/2} e^{in(x-x_j)} = \frac{1}{N+1}D_{N/2}(x-x_j), \tag{41}$$

which coincides with (36).

The first-order derivative of the discrete Fourier series evaluated at the evenly spaced nodes (33) is

$$\frac{du_{N/2}(x_p)}{dx} = \sum_{j=0}^{N} u(x_j)\underbrace{\frac{dh_j(x_p)}{dx}}_{D_{pj}} \tag{42}$$

where

$$D_{pj} = \begin{cases} \dfrac{(-1)^{p+j}}{2\sin\left(\dfrac{x_p-x_j}{2}\right)} & p\neq j \\ 0 & p = j \end{cases} \tag{43}$$

is the first-order differentiation matrix for the odd expansion. Higher order derivatives can be expressed as

$$\frac{d^m u_{N/2}(x_p)}{dx^m} = \sum_{j=0}^{N} D^{(m)}_{pj} u(x_j) \tag{44}$$

where the m order differentiation matrix $\boldsymbol{D}^{(m)}$ in this case is exactly the product of m first-order differentiation matrices $\boldsymbol{D}$

$$\boldsymbol{D}^{(m)} = \underbrace{\boldsymbol{D}\cdots\boldsymbol{D}}_{m \text{ times}}. \tag{45}$$

[5]To prove (40) we notice that

$$1 + 2\sum_{k=1}^{N}\cos(kx) = 1 + \sum_{k=1}^{N}\left(e^{ikx} + e^{-ikx}\right) = \sum_{k=0}^{N}\left(e^{ix}\right)^k + \sum_{k=0}^{N}\left(e^{-ix}\right)^k - 1. \tag{38}$$

Summing up the geometric series yields

$$\sum_{k=0}^{N}\left(e^{ix}\right)^k = \frac{e^{ix(N+1)}-1}{e^{ix}-1}, \qquad \sum_{k=0}^{N}\left(e^{-ix}\right)^k = \frac{e^{-ix(N+1)}-1}{e^{-ix}-1}. \tag{39}$$

A substitution of (39) into (38) yields, after some algebra, equation (40).

Fourier-Galerkin and Fourier-collocation methods

In this section we present Fourier-Galerkin and Fourier-collocation methods for the solution of partial differential equations. As in the previous chapter we restrict ourselves to one-dimensional problems with periodic boundary conditions on $[0, 2\pi]$. To this end, consider the nonlinear initial/boundary value problem

$$\begin{cases} \dfrac{\partial U(x,t)}{\partial t} = N(U)(x,t) + f(x,t) \qquad x \in [0,2\pi] \\ U(x,0) = U_0(x) \\ \text{Periodic B.C.} \end{cases} \tag{46}$$

where N is a nonlinear operator on U which may also depend on x and t. Examples of the $N(U)$ are

$$N(U) = \begin{cases} -a(x)\dfrac{\partial U}{\partial x} & \text{(linear advection)} \\ -U\dfrac{\partial U}{\partial x} + (2+\cos(x))\dfrac{\partial^2 U}{\partial x^2} & \text{(nonlinear advection/diffusion)} \\ -U\dfrac{\partial U}{\partial x} - \dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial^4 U}{\partial x^4} & \text{(Kuramoto-Sivashinsky operator)} \end{cases} \tag{47}$$

Fourier-Galerkin method

In the Fourier-Galerkin method, we seek solutions to (46) of the form

$$u_{N/2}(x,t) = \sum_{|n|\leq N/2} u_n(t) e^{inx}, \tag{48}$$

where N is even and $u_n(t)$ are complex-valued functions of time. A substitution of (48) into (46) yields the residual

$$R_{N/2}(x,t) = \frac{\partial u_{N/2}}{\partial t} - N(u_{N/2}) - f(x,t). \tag{49}$$

In the Galerkin method we impose that the residual is orthogonal in the sense of $L^2([0, 2\pi])$ to the linear space in which we seek for a solution, i.e.,

$$B_{N/2} = \operatorname{span}\left\{ e^{inx} \, : \, |n| \leq \frac{N}{2} \right\}. \tag{50}$$

This yields the following system of $N + 1$ ODEs in the $N + 1$ coefficients $u_n(t)$

$$\left\langle R_{N/2}(x,t), e^{-inx} \right\rangle = \int_0^{2\pi} R_{N/2}(x,t) e^{-inx} dx = 0 \qquad |n| \leq \frac{N}{2}. \tag{51}$$

Example 1: Consider the linear advection-diffusion equation

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial U}{\partial x} + \dfrac{\partial^2 U}{\partial x^2} \qquad x \in [0,2\pi] \\ U(x,0) = U_0(x) \\ \text{Periodic B.C.} \end{cases} \tag{52}$$

A substitution of (48) into (52) yields the residual

$$R_{N/2}(x,t) = \sum_{|p|\leq N/2} \left(\frac{du_p(t)}{dt} - ipu_p(t) + p^2 u_p(t) \right) e^{ipx}. \tag{53}$$

Imposing the orthogonality conditions (51) yields

$$\sum_{|p|\leq N/2} \left(\frac{du_p(t)}{dt} - ipu_p(t) + p^2 u_p(t) \right) \int_0^{2\pi} e^{i(p-n)x} dx = 0, \qquad |n| \leq \frac{N}{2}. \tag{54}$$

The only nonzero integrals are for $p = n$. This yields the ODE system

$$\begin{cases} \dfrac{du_n(t)}{dt} = inu_n(t) - n^2 u_n(t) \\ u_n(0) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} U_0(x) e^{-inx} dx \end{cases} \qquad |n| \leq \frac{N}{2}. \tag{55}$$

Note that the ODE system (55) is linear and decoupled. Hence it can be easily solved analytically as

$$u_n(t) = u_n(0) e^{-n^2 t} e^{int}. \tag{56}$$

Example 2: Consider the following advection-diffusion equation

$$\frac{\partial U}{\partial t} = \cos(x) \frac{\partial U}{\partial x} + [2 + \sin(x)] \frac{\partial^2 U}{\partial x^2}. \tag{57}$$

A substitution of (48) into (57) yields the residual

$$R_{N/2}(x,t) = \sum_{|p|\leq N/2} e^{ipx} \left[\frac{du_p}{dt} - \frac{e^{ix} + e^{-ix}}{2} ipu_p + \left(2 + \frac{e^{ix} - e^{-ix}}{2i} \right) p^2 u_p \right] \tag{58}$$

Projecting the residual onto (50) yields

$$\begin{aligned} \frac{du_n}{dt} =& -2n^2 u_n + \frac{1}{2\pi} \sum_{|p|\leq N/2} \left(ipu_p \int_0^{2\pi} \frac{e^{i(1-n+p)x} + e^{i(-1-n+p)x}}{2} dx - p^2 u_p \int_0^{2\pi} \frac{e^{i(1-n+p)x} - e^{i(-1-n+p)x}}{2i} dx \right) \\ =& -2n^2 u_n + i \frac{(n-1)u_{n-1} + i(n+1)u_{n+1}}{2} - \frac{(n-1)^2 u_{n-1} - (n+1)^2 u_{n+1}}{2i} \qquad |n| \leq \frac{N}{2}. \end{aligned} \tag{59}$$

As before, this system of ODEs is supplemented with the initial condition

$$u_n(0) = \frac{1}{2\pi} \int_0^{2\pi} U_0(x) e^{-inx} dx. \tag{60}$$

Fourier-collocation method

We can circumvent the need for evaluating the inner products integrals by using quadrature formulas. This is equivalent to using the interpolating operator (discrete Fourier series) instead of the orthogonal projection operator, and is called the Fourier–collocation method, or Fourier pseudo-spectral method. To form the Fourier–collocation method we require that the residual of the PDE vanishes identically on some set of grid points, e.g., the even grid (17)

$$x_j = \frac{2\pi}{N} j \qquad j = 0, \ldots, N-1. \tag{61}$$

In the Fourier-collocation method we look for a solution expressed as a trigonometric polynomial via discrete Fourier series expansion. If we use the even expansion we have

$$u_{N/2}(x,t) = \sum_{j=0}^{N-1} u_{N/2}(x_j,t) g_j(x), \tag{62}$$

where $g_j(x)$ is the Lagrange interpolation polynomial for an even number of points (see (25)). A substitution of (62) into (46) yields the residual

$$R_{N/2}(x,t) = \frac{\partial u_{N/2}}{\partial t} - N(u_{N/2}) - f(x,t). \tag{63}$$

In the Fourier collocation method we impose that the residual is zero at the N collocation nodes $\{x_0,\ldots,x_{N-1}\}$, i.e.,

$$R_{N/2}(x_j,t) = 0 \qquad j = 0,\ldots,N-1 \tag{64}$$

This yields a system of N ODEs in N unknowns

$$\{u_{N/2}(x_0,t),\ldots,u_{N/2}(x_{N-1},t)\}. \tag{65}$$

Example 1: Consider the linear advection-diffusion equation (52). A substitution of (62) into (52) and collocation at $\{x_0,\ldots,x_{N-1}\}$ yields[6]

$$\begin{cases} \dfrac{u_{N/2}(x_p,t)}{dt} = \displaystyle\sum_{j=0}^{N-1} \left(D_{pj} + D^{(2)}_{pj} \right) u_{N/2}(x_j,t) \\ \\ u_{N/2}(x_p,0) = U_0(x_p) \end{cases} \tag{67}$$

where D_{pj} and $D^{(2)}_{pj}$ are the first- and second-order differentiation matrices defined in (31) and (32), respectively.

Example 2: Consider the advection-diffusion problem (57). By substituting (62) into (57) and setting the residual $R_{N/2}(x,t)$ to zero at collocation nodes $\{x_0,\ldots,x_{N-1}\}$ yields

$$\begin{cases} \dfrac{u_{N/2}(x_p,t)}{dt} = \cos(x_p) \displaystyle\sum_{j=0}^{N-1} D_{pj} u_{N/2}(x_j,t) + (2+\sin(x_p)) \sum_{j=0}^{N-1} D^{(2)}_{pj} u_{N/2}(x_j,t) \\ \\ u_{N/2}(x_p,0) = U_0(x_p) \end{cases} \tag{68}$$

Example 3: Consider the highly nonlinear advection problem

$$\frac{\partial U}{\partial t} = e^{U(x,t)} \frac{\partial U}{\partial x}. \tag{69}$$

[6] To derive (67) we notice that

$$R_{N/2}(x,t) = \sum_{j=1}^{N-1} \frac{du_{N/2}(x_j,t)}{dt} g_j(x) - \sum_{j=1}^{N-1} u_{N/2}(x_j,t) \frac{dg_j(x)}{dx} - \sum_{j=1}^{N-1} u_{N/2}(x_j,t) \frac{d^2 g_j(x)}{dx^2}. \tag{66}$$

Setting the residual equal to zero at x_p ($p = 1,\ldots,N-1$) and recalling the definition of differentiation matrices (31) and (32) immediately yields (67).

The Fourier collocation method for this problem yields the ODE system

$$\begin{cases} \dfrac{u_{N/2}(x_p,t)}{dt} = e^{u_{N/2}(x_p,t)} \displaystyle\sum_{j=0}^{N-1} D_{pj} u_{N/2}(x_j,t) \\ u_{N/2}(x_p,0) = U_0(x_p) \end{cases} \tag{70}$$

Note that the Fourier-Galerkin system for (69) is much harder to deal with.

References

[1] J. S. Hesthaven, S. Gottlieb, and D. Gottlieb. *Spectral methods for time-dependent problems*, volume 21 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2007.