Boundary value problems for ODEs

A boundary value problem (BVP) for an ODE is a problem in which we set conditions on the solution to the ODE corresponding to different values in the independent variable. Such conditions can be on the solution, on the derivatives of the solution, or more general conditions. Perhaps the simplest boundary value problem for an ODE is[1]

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} = f(x) \qquad x \in [0,1] \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \tag{2}$$

in which we set conditions on the value of the solution at $x = 0$ and $x = 1$. Such conditions are called *Dirichlet boundary conditions*. The general solution to (2) can be written as

$$u(x) = c_1 + c_2 x + \int_0^x F(s)ds \qquad \text{where} \qquad F(s) = \int_0^s f(y)dy. \tag{3}$$

By using integration by parts

$$\int_0^x F(s)ds = [sF(s)ds]_{s=0}^{s=x} - \int_0^x sf(s)ds = \int_0^x (x-s)f(s)ds. \tag{4}$$

Substituting this expression into (3) yields

$$u(x) = c_1 + c_2 x + \int_0^x (x-s)f(s)ds. \tag{5}$$

At this point we enforce the boundary conditions to obtain

$$\alpha = c_1 \qquad \beta = c_1 + c_2 + \int_0^1 (1-s)f(s)ds, \tag{6}$$

which gives the following unique solution to (2)

$$u(x) = \alpha + x\left(\beta - \alpha - \int_0^1 (1-s)f(s)ds\right) + \int_0^x (x-s)f(s)ds. \tag{7}$$

Lemma 1. For every $f \in C^0([0,1])$ there exists a unique solution $u \in C^2([0,1])$ to the boundary value problem (2). Moreover, if $f \in C^k([0,1])$ then $u \in C^{k+2}([0,1])$.

Green function and maximum principle. The solution (7) corresponding to zero Dirichlet conditions can be conveniently written in terms of an integral involving a Green function. Setting $\alpha = \beta = 0$ in (7)

[1]From a physical viewpoint, the BVP (2) defines a steady state heat conduction problem in a one-dimensional slab with uniform conductivity, heat generation, and fixed temperature conditions at the boundary. In fact (2) can be derived from the Fourier equation [1]

$$\frac{\partial u}{\partial t} = \frac{\lambda}{\rho c_p}\nabla^2 u + \frac{1}{\lambda} f(\boldsymbol{x}). \tag{1}$$

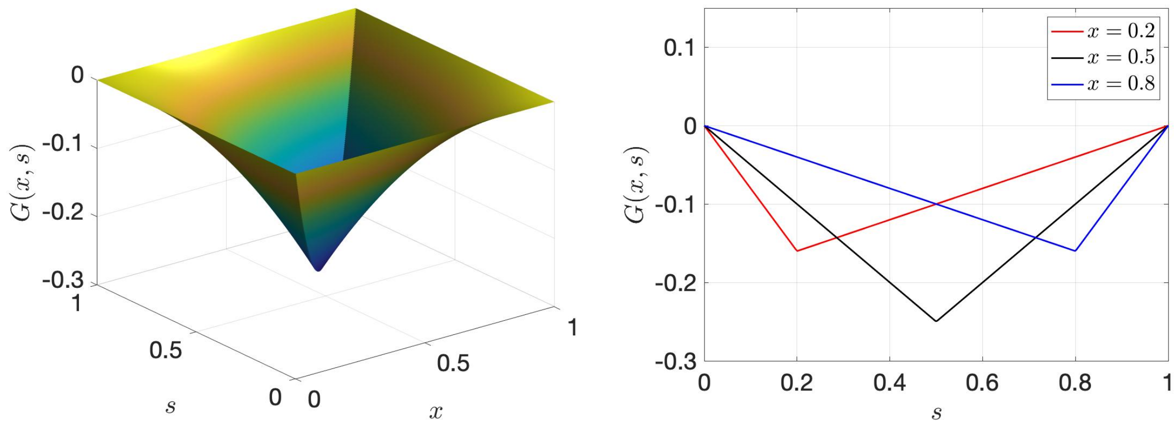

Figure 1: Green function $G(s, x)$ defined in equations (8)-(9).

yields

$$\begin{aligned} u(x) &= -\,x \int_0^1 (1-s) f(s) ds + \int_0^x (x-s) f(s) ds \\ &= \int_0^x \left[(x-s) - x(1-s)\right] f(s) ds - x \int_x^1 (1-s) f(s) ds \\ &= \int_0^x s(x-1) f(s) ds + \int_x^1 x(s-1) f(s) ds \\ &= \int_0^1 G(x,s) f(s) ds, \end{aligned} \tag{8}$$

where we defined

$$G(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases} \qquad \text{(Green function)}. \tag{9}$$

The Green function $G(x, s)$ is the kernel of the integral operator (8), and it represents the “response” of the system corresponding to any forcing function $f(x)$. The Green function satisfies (in a distributional sense, and for all $s \in [0, 1]$) the boundary value problem

$$\begin{cases} \dfrac{d^2 G(x,s)}{dx^2} = \delta(x-s) \\ G(0,s) = 0 \\ G(1,s) = 0 \end{cases} \tag{10}$$

With the Green function available, it is straightforward to obtain the following bound for (8)

$$\|u\|_\infty \le \frac{1}{8} \|f\|_\infty \qquad \text{(maximum principle)}, \tag{11}$$

where $\|\cdot\|_\infty$ here denotes the uniform norm of a function, i.e.,

$$\|u\|_\infty = \sup_{x\in[0,1]} |u(x)|, \qquad \|f\|_\infty = \sup_{x\in[0,1]} |f(x)|. \tag{12}$$

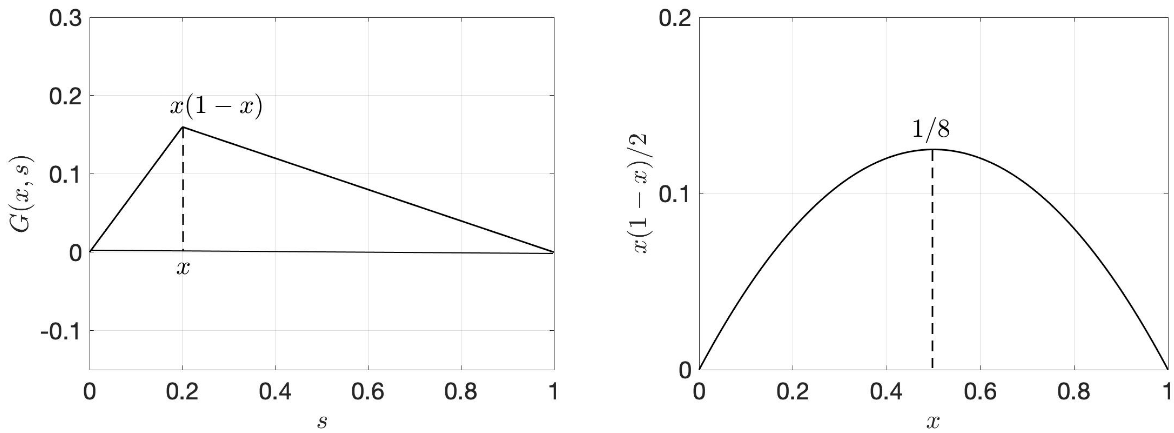

Figure 2: Evaluation of the integral appearing in (13).

The inequality (11), states that the solution of the boundary value problem (2) with homogeneous Dirichlet conditions ($\alpha = \beta = 0$) is always smaller that $1/8$ of the maximum value of $f(x)$ in the domain $[0, 1]$. To prove (11) we observe that

$$|u(x)| \leq \int_0^1 |G(x,s)|\,|f(s)|\,ds \leq \|f\|_\infty \int_0^1 |G(x,s)|\,ds. \tag{13}$$

The Green function $G(x, s)$ is always negative, and for each fixed it is the union of two triangular functions joining at the point $(x, x(1, x))$ (see Figure 2). Therefore,

$$\int_0^1 |G(x,s)|\,ds = x\frac{x(1-x)}{2} + (1-x)\frac{x(1-x)}{2} = \frac{x(1-x)}{2}. \tag{14}$$

Substituting this result into (13) yiels

$$|u(x)| \leq \|f\|_\infty \frac{x(1-x)}{2}. \tag{15}$$

Finally, by taking the maximum over all $x \in [0, 1]$ we obtain[2]

$$\max_{x\in[0,1]} |u(x)| \leq \|f\|_\infty \max_{x\in[0,1]} \frac{x(1-x)}{2} = \frac{1}{8}\|f\|_\infty \tag{17}$$

which coincides with (11).

Ill-posed linear boundary value problems. If we replace the Dirichlet boundary conditions in (2) with two *Neumann boundary conditions* (i.e., we set the value of the derivative of $u(x)$ at $x = 0$ and $x = 1$ instead of the value of the function) then the problem can have either no solution or an infinite number of

[2]The maximum of the function $x(x-1)/2$ is $1/8$ and it is attained at $x = 1/2$ (see Figure 2), i.e., we have

$$\max_{x\in[0,1]} \int_0^1 |G(x,s)|\,ds = \max_{x\in[0,1]} \frac{x(1-x)}{2} = \frac{1}{8}. \tag{16}$$

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solutions. To show this, let us consider the BVP

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} = f(x) \qquad x \in [0,1] \\ \dfrac{du(0)}{dx} = \alpha \\ \dfrac{du(1)}{dx} = \beta \end{cases} \tag{18}$$

By integrating the ODE once, we obtain

$$\frac{du(x)}{dx} = c_1 + \int_0^x f(s)ds \tag{19}$$

which shows that the derivative of u depends only on one arbitrary constant of integration. Clearly, we do not have enough degrees of freedom to satisfy (in general) both boundary conditions in (18). By enforcing $du(0)/dx = \alpha$ we obtain $c_1 = \alpha$, i.e.,

$$\frac{du(x)}{dx} = \alpha + \int_0^x f(s)ds. \tag{20}$$

If we now try to enforce $du(1)/dx = \beta$ in (20) we obtain the equation

$$\beta - \alpha = \int_0^1 f(s)ds. \tag{21}$$

If $f(x)$ satisfies (21) then the problem (18) has an infinite number of solutions. In fact, by integrating (20) we see that there exits a one-parameter family of solutions (with parameter c_2) of the form

$$u(x) = c_2 + \alpha x + \int_0^x \left(\int_0^y f(s)ds \right) dy. \tag{22}$$

Clearly, the solution (22) satisfies (18) for all $c_2 \in \mathbb{R}$, provided (21) holds. On the other hand, if $f(x)$ does not satisfy (21) then the boundary value problem (18) has no solution.

Exercise: By using a physical argument based on the interpretation of (18) as a model of heat conduction in a one-dimensional slab with heat generation, justify the infinite multiplicity of solutions or the lack of a solution.

Example: It is straightforward to show that the linear BVP

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 0, \qquad y(\pi) = 0. \tag{23}$$

has no solution. In fact, the flow generated by the corresponding first-order system is a center. There are in principle infinite trajectories that start from $y = 0$ and end at $y = 0$. None of them though makes the trip exactly in π time units.

Ill-posed nonlinear boundary value problems. Next, consider a nonlinear boundary value problem of the form

$$\begin{cases} \dfrac{d^2y}{dt^2} = f\left(\dfrac{dy}{dt}, y, t\right) \qquad t \in [0,T] \\ y(0) = \alpha \\ y(T) = \beta \end{cases} \tag{24}$$

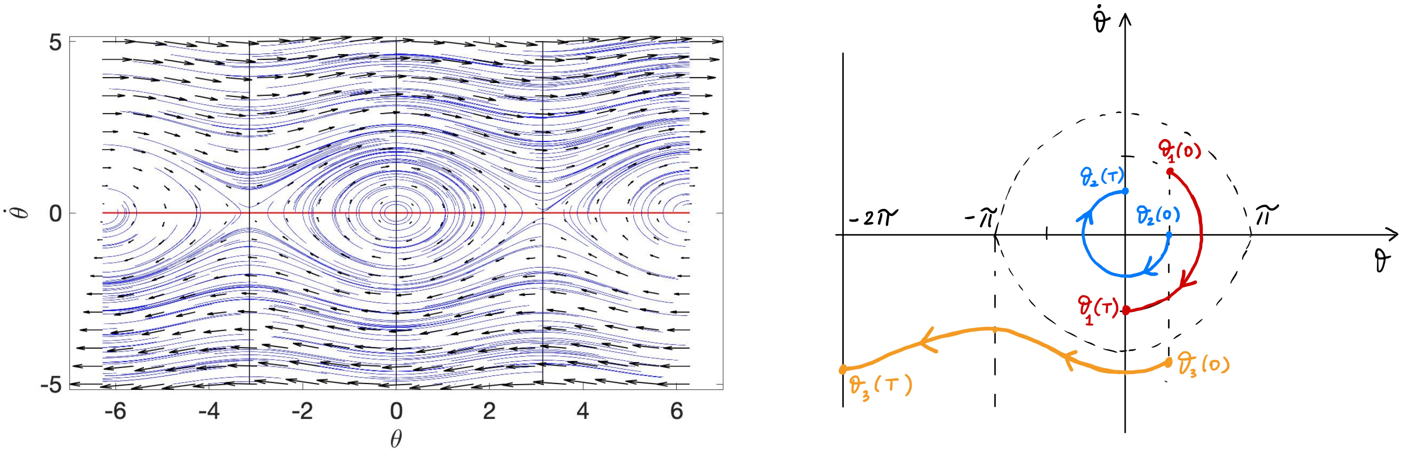

Figure 3: Phase portrait of the pendulum equation $\ddot{\theta} = -\sin(\theta)$ and sketch of two solutions (ϑ_1 and ϑ_2) of the BVP (25). The third solution technically does not satisfy $\theta(T) = 0$ but rather $\theta(T) = -2\pi$ which is physically equivalent, but mathematically different.

It is easy to show by a simple physical example that this problem can have an infinite number of solutions (all of which make sense). To this end, consider the pendulum equations

$$\begin{cases} \dfrac{d^2\theta}{dt^2} = -\sin(\theta) \\ \theta(0) = \dfrac{\pi}{2} \\ \theta(T) = 0 \end{cases} \tag{25}$$

where T is the time that it takes to the pendulum to reach the vertical position after swinging from right to left only once from a zero velocity initial condition. It is clear that there are multiple solutions to this problem. In Figure 3 we sketch two of such initial velocities, and corresponding trajectories.

Exercise: What's the motion of the pendulum corresponding to the paths $(\theta_i, \dot{\theta}_i)$ sketched in Figure 3 for $i = 1, 2, 3$? Interpret the infinite (countable) number of solutions of (25) physically. How many solutions are there within the initial velocity interval $\dot{\theta}(0) \in [-v, v]$, for a given v?

Existence and uniqueness of solutions. There is no general theory for existence and uniqueness of the solution to nonlinear two-point boundary value problems with arbitrary boundary conditions. However, a lot can be said in specific cases. For example, it is straightforward to show that the two-point boundary value problems for the linear system of ODEs

$$\frac{d^2\boldsymbol{y}}{dt^2} = \boldsymbol{A}\boldsymbol{y}, \tag{26}$$

with diagonalizable $\boldsymbol{A}$ and Dirichlet boundary conditions $\boldsymbol{y}(0) = \boldsymbol{\alpha}$ and $\boldsymbol{y}(1) = \boldsymbol{\beta}$ has a unique solution. In fact, upon definition of $\boldsymbol{z} = d\boldsymbol{y}/dt$ we can write (26) as

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix}. \tag{27}$$

Let $\boldsymbol{P}$ and $\boldsymbol{\Lambda}$ be the matrix of eigenvectors and the diagonal matrix of eigenvalues of $\boldsymbol{A}$, i.e.,

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}, \tag{28}$$

and consider the transformation induced by the invertible block matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix}. \tag{29}$$

Clearly,

$$\underbrace{\begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{\Lambda} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}^{-1} \end{bmatrix}}_{\boldsymbol{H}^{-1}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{30}$$

By applying $\boldsymbol{H}^{-1}$ to the system (27) we obtain

$$\frac{d}{dt}\begin{bmatrix} \tilde{\boldsymbol{z}} \\ \tilde{\boldsymbol{y}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{\Lambda} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{C}} \begin{bmatrix} \tilde{\boldsymbol{z}} \\ \tilde{\boldsymbol{y}} \end{bmatrix}, \qquad \text{where} \qquad \begin{bmatrix} \tilde{\boldsymbol{z}} \\ \tilde{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} \tag{31}$$

The solution to this ODE (treated as initial value problem with unknown $\tilde{\boldsymbol{z}}(0)$) is

$$\begin{bmatrix} \tilde{\boldsymbol{z}}(t) \\ \tilde{\boldsymbol{y}}(t) \end{bmatrix} = e^{t\boldsymbol{C}} \begin{bmatrix} \tilde{\boldsymbol{z}}(0) \\ \tilde{\boldsymbol{y}}(0) \end{bmatrix}. \tag{32}$$

Setting the boundary conditions $\boldsymbol{y}(0) = \boldsymbol{\alpha}$ and $\boldsymbol{y}(1) = \boldsymbol{\beta}$ yields

$$\begin{bmatrix} \tilde{\boldsymbol{z}}(1) \\ \boldsymbol{P}^{-1}\boldsymbol{\beta} \end{bmatrix} = e^{\boldsymbol{C}} \begin{bmatrix} \tilde{\boldsymbol{z}}(0) \\ \boldsymbol{P}^{-1}\boldsymbol{\alpha} \end{bmatrix}. \tag{33}$$

The exponential matrix $e^{\boldsymbol{C}}$ has the following structure

$$e^{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_2 \\ \boldsymbol{D}_3 & \boldsymbol{D}_1 \end{bmatrix}, \tag{34}$$

where $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$ are diagonal matrices. Moreover, $\boldsymbol{D}_1$ and $\boldsymbol{D}_3$ are invertible. Substituting (34) into (33) gives

$$\begin{bmatrix} \tilde{\boldsymbol{z}}(1) \\ \boldsymbol{P}^{-1}\boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{D}_2 \\ \boldsymbol{D}_3 & \boldsymbol{D}_1 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{z}}(0) \\ \boldsymbol{P}^{-1}\boldsymbol{\alpha} \end{bmatrix}. \tag{35}$$

This equation allows us to determine $\tilde{\boldsymbol{z}}(0)$ uniquely for any given $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ In fact, the second equation in (35) can be written as

$$\boldsymbol{D}_3\tilde{\boldsymbol{z}}(0) = \boldsymbol{P}^{-1}\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{P}^{-1}\boldsymbol{\alpha} \qquad \Leftrightarrow \qquad \tilde{\boldsymbol{z}}(0) = \boldsymbol{D}_3^{-1}\boldsymbol{P}^{-1}\left(\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{\alpha}\right). \tag{36}$$

Hence, we proved that for every given $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ there exists a unique initial state

$$\begin{bmatrix} \widetilde{\boldsymbol{y}}(0) \\ \widetilde{\boldsymbol{z}}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}^{-1}\boldsymbol{\alpha} \\ \boldsymbol{D}_3^{-1}\boldsymbol{P}^{-1}\left(\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{\alpha}\right) \end{bmatrix}. \tag{37}$$

By leveraging the existence and uniqueness of solutions to the initial value problem (32) we conclude that the two-point boundary value problem for the ODE (26) with Dirichlet boundary conditions has a unique solution.

Remark: If we drop the assumption of diagonalizability of $\boldsymbol{A}$ and replace the diagonal matrix $\boldsymbol{\Lambda}$ with its block diagonal Jordan form $\boldsymbol{J}$, then the ODE (26) with Dirichlet boundary conditions still has a unique solution. In fact, in this case the matrix exponential $e^{\boldsymbol{C}}$ is still a block matrix in the form (34), but with upper triangular $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$. Moreover, $\boldsymbol{D}_1$ and $\boldsymbol{D}_3$ are invertible. Hence (36) still holds.

General form of two-point boundary value problems. A two-point boundary value problem for a system of n-dimensional nonlinear ODEs can be written in the general form

$$\begin{cases} \dfrac{d\boldsymbol{y}}{dt} = \boldsymbol{f}\left(\boldsymbol{y}, t\right) \qquad t \in [0, T] \\ \boldsymbol{g}(\boldsymbol{y}(0), \boldsymbol{y}(T)) = \boldsymbol{0} \end{cases} \tag{38}$$

where $\boldsymbol{g} \in \mathbb{R}^n$ is nonlinear function. All two-point boundary value problem we studied so far can be written in this form, provided we define appropriate phase variables $\boldsymbol{y}$, the right hand side $\boldsymbol{f}(\boldsymbol{y}, t)$, and the boundary function $\boldsymbol{g}$.

References

[1] D. W. Hahn and M. N. Özisik. *Heat Conduction.* Wiley, third edition, 2012.