Boundary value problems for ODEs

A boundary value problem (BVP) for an ODE is a problem in which we set conditions on the solution to the ODE at different values of the independent variable. Such conditions can be on the solution itself, on the derivatives of the solution, or more general conditions involving nonlinear functions of the solution. Perhaps, the simplest boundary value problem for an ODE is[1]

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} = f(x) \qquad x \in [0,1] \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \tag{2}$$

in which we set conditions on the value of the solution at $x = 0$ and $x = 1$. Such conditions are called *Dirichlet boundary conditions*. The general solution to (2) can be written as

$$u(x) = c_1 + c_2 x + \int_0^x F(s)ds \qquad \text{where} \qquad F(s) = \int_0^s f(y)dy. \tag{3}$$

By using integration by parts

$$\int_0^x F(s)ds = [sF(s)ds]_{s=0}^{s=x} - \int_0^x sf(s)ds = \int_0^x (x-s)f(s)ds. \tag{4}$$

Substituting this expression into (3) yields

$$u(x) = c_1 + c_2 x + \int_0^x (x-s)f(s)ds. \tag{5}$$

At this point we enforce the boundary conditions to obtain

$$\alpha = c_1 \qquad \beta = c_1 + c_2 + \int_0^1 (1-s)f(s)ds, \tag{6}$$

which gives the following unique solution to (2)

$$u(x) = \alpha + x\left(\beta - \alpha - \int_0^1 (1-s)f(s)ds\right) + \int_0^x (x-s)f(s)ds. \tag{7}$$

Lemma 1. For every $f \in C^0([0,1])$ there exists a unique solution $u \in C^2([0,1])$ to the boundary value problem (2). Moreover, if $f \in C^k([0,1])$ then $u \in C^{k+2}([0,1])$.

Green functions and maximum principle. The solution (7) corresponding to zero Dirichlet conditions ($\alpha = \beta = 0$) can be conveniently written as

$$u(x) = \int_0^1 G(x,s)f(s)ds, \tag{8}$$

i.e., in terms of an integral operator with appropriate kernel $G(x,s)$. As we shall see hereafter $G(x,s)$ is the Green function of the problem.

[1] From a physical viewpoint, the BVP (2) defines a steady state heat conduction problem in a one-dimensional slab with uniform conductivity, heat generation, and fixed temperature conditions at the boundary. In fact (2) can be derived from the Fourier equation [1]

$$\frac{\partial u}{\partial t} = \frac{\lambda}{\rho c_p}\nabla^2 u + \frac{1}{\lambda} f(\boldsymbol{x}). \tag{1}$$

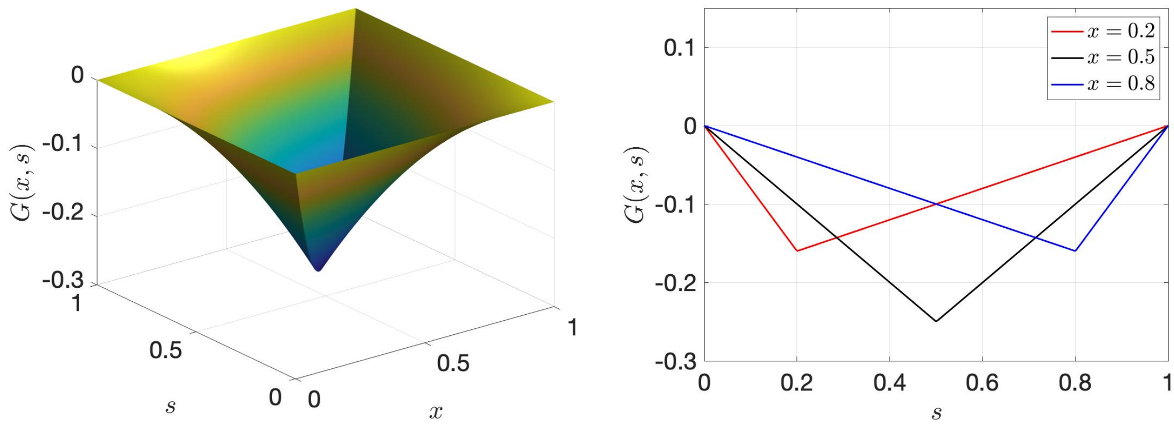

Figure 1: Green function $G(s, x)$ defined in equations (9)-(10).

Setting $\alpha = \beta = 0$ in (7) yields

$$\begin{aligned} u(x) &= - x \int_0^1 (1-s) f(s) ds + \int_0^x (x-s) f(s) ds \\ &= \int_0^x \left[(x-s) - x(1-s) \right] f(s) ds - x \int_x^1 (1-s) f(s) ds \\ &= \int_0^x s(x-1) f(s) ds + \int_x^1 x(s-1) f(s) ds \\ &= \int_0^1 G(x,s) f(s) ds, \end{aligned} \tag{9}$$

where we defined

$$G(x,s) = \begin{cases} s(1-x) & 0 \leq s \leq x \\ x(1-s) & x \leq s \leq 1 \end{cases} \qquad \text{(Green function).} \tag{10}$$

The Green function $G(x, s)$ is the kernel of the integral operator (9), and it represents the "response" of the system corresponding to any forcing function $f(x)$. The Green function satisfies (in a distributional sense, and for all $s \in [0, 1]$) the boundary value problem

$$\begin{cases} \dfrac{d^2 G(x,s)}{dx^2} = \delta(x-s) \\ G(0,s) = 0 \\ G(1,s) = 0 \end{cases} \tag{11}$$

With the Green function available, it is straightforward to obtain the following bound for (9)

$$\left\| u \right\|_\infty \leq \frac{1}{8} \left\| f \right\|_\infty \qquad \text{(maximum principle),} \tag{12}$$

where $\left\| \cdot \right\|_\infty$ here denotes the uniform norm of a function, i.e.,

$$\left\| u \right\|_\infty = \max_{x \in [0,1]} \left| u(x) \right|, \qquad \left\| f \right\|_\infty = \max_{x \in [0,1]} \left| f(x) \right|. \tag{13}$$

The inequality (12), states that the solution of the boundary value problem (2) with homogeneous Dirichlet conditions ($\alpha = \beta = 0$) is always smaller that 1/8 of the maximum value of $f(x)$ in the domain $[0, 1]$. To

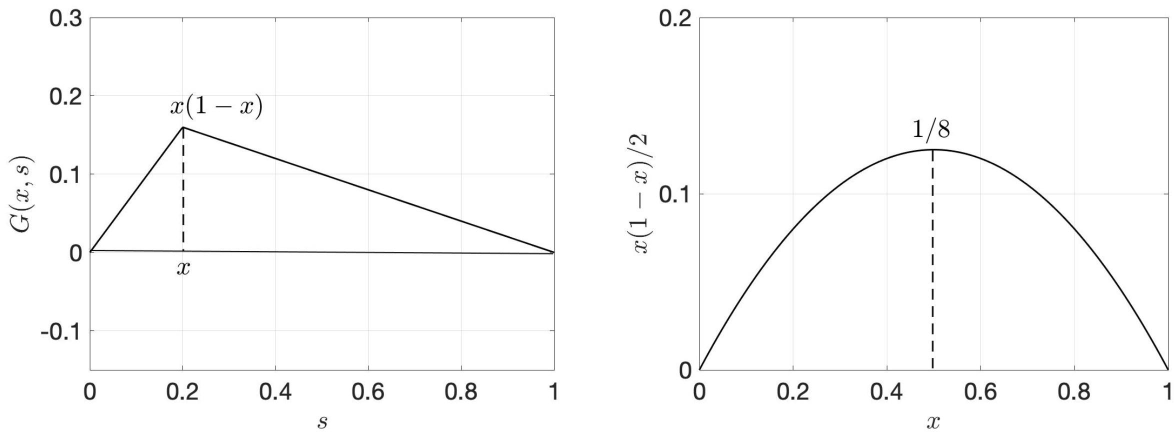

Figure 2: Evaluation of the integral appearing in (14).

prove (12) we observe that

$$|u(x)| \leq \int_0^1 |G(x,s)|\,|f(s)|\,ds \leq \|f\|_\infty \int_0^1 |G(x,s)|\,ds. \tag{14}$$

The Green function $G(x,s)$ is always negative, and for each fixed it is the union of two triangular functions joining at the point $(x, x(1,x))$ (see Figure 2). Therefore,

$$\int_0^1 |G(x,s)|\,ds = x\frac{x(1-x)}{2} + (1-x)\frac{x(1-x)}{2} = \frac{x(1-x)}{2}. \tag{15}$$

Substituting this result into (14) yiels

$$|u(x)| \leq \|f\|_\infty \frac{x(1-x)}{2}. \tag{16}$$

Finally, by taking the maximum over all $x \in [0,1]$ we obtain[2]

$$\max_{x\in[0,1]} |u(x)| \leq \|f\|_\infty \max_{x\in[0,1]} \frac{x(1-x)}{2} = \frac{1}{8}\|f\|_\infty \tag{18}$$

which coincides with (12).

General form of two-point boundary value problems for ODEs. A two-point boundary value problem for a system of n-dimensional nonlinear (normal) ODEs can always be written in the general form

$$\begin{cases} \dfrac{d\boldsymbol{y}}{dt} = \boldsymbol{f}(\boldsymbol{y}, t) & t \in [0,T] \\ \boldsymbol{g}(\boldsymbol{y}(0), \boldsymbol{y}(T)) = \boldsymbol{0} \end{cases} \tag{19}$$

where $\boldsymbol{g} \in \mathbb{R}^n$ is nonlinear function.

[2] The maximum of the function $x(x-1)/2$ is $1/8$ and it is attained at $x = 1/2$ (see Figure 2), i.e., we have

$$\max_{x\in[0,1]} \int_0^1 |G(x,s)|\,ds = \max_{x\in[0,1]} \frac{x(1-x)}{2} = \frac{1}{8}. \tag{17}$$

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Existence and uniqueness of solutions. There is no general theory that guarantees existence and uniqueness of the solution to the two-point boundary value problem (19). And for good reasons! The solution may not be unique or may not exist at all. However, detailed analysis can often be provided for particular problems with particular boundary conditions. For instance, it is rather easy to show that boundary value problems for second-order linear ordinary differential equations with Dirichlet boundary conditions have unique solutions. This is summarized in the following theorem.

Theorem 1. Let $\boldsymbol{A}$ be a $n \times n$ matrix. There exists a unique solution to the two-point eigenvalue problem

$$\begin{cases} \dfrac{d^2\boldsymbol{y}}{dt^2} = \boldsymbol{A}\boldsymbol{y} \qquad t \in [0, T] \\ \boldsymbol{y}(0) = \boldsymbol{\alpha} \\ \boldsymbol{y}(T) = \boldsymbol{\beta} \end{cases} \tag{20}$$

The proof of this theorem is given in Appendix A. Various generalizations to other types of second-order linear ODEs, e.g.,

$$\frac{d^2\boldsymbol{y}}{dt^2} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}\frac{d\boldsymbol{y}}{dt} + \boldsymbol{f}(t) \tag{21}$$

or higher-order ODEs are possible. In any case, the study of existence and uniqueness of the solution to a particular type of two-point boundary value problem must be carried out on an individual basis. In fact, as we shall see hereafter, if we consider other types of boundary conditions, or if the ODE is nonlinear problems then the BVP can have infinite solutions or no solution at all!

Ill-posed linear boundary value problems

If we replace the Dirichlet boundary conditions in (2) with two *Neumann boundary conditions* (i.e., we set the value of the derivative of $u(x)$ at $x = 0$ and $x = 1$ instead of the value of the function) then the problem can have either no solution or an infinite number of solutions. To show this, let us consider the BVP[3]

$$\begin{cases} \dfrac{d^2u(x)}{dx^2} = f(x) \qquad x \in [0, 1] \\ \dfrac{du(0)}{dx} = \alpha \\ \dfrac{du(1)}{dx} = \beta \end{cases} \tag{22}$$

By integrating the ODE once, we obtain

$$\frac{du(x)}{dx} = c_1 + \int_0^x f(s)ds \tag{23}$$

which shows that the derivative of u depends only on one arbitrary constant of integration. Clearly, we do not have enough degrees of freedom to satisfy (in general) both boundary conditions in (22). By enforcing $du(0)/dx = \alpha$ we obtain $c_1 = \alpha$, i.e.,

$$\frac{du(x)}{dx} = \alpha + \int_0^x f(s)ds. \tag{24}$$

[3]The Boundary value problem defined in equation (2) describes the temperature propagation (by pure heat condition) within a homogeneous one-dimensional slab of width one subject to uniform Neumann boundary conditions, uniform thermal conductivity, and heat generation that depends on x.

If we now try to enforce $du(1)/dx = \beta$ in (24) we obtain the equation

$$\beta - \alpha = \int_0^1 f(s)ds. \tag{25}$$

If $f(x)$ satisfies (25) then the problem (22) has an infinite number of solutions. In fact, by integrating (24) we see that there exits a one-parameter family of solutions (with parameter c_2) of the form

$$u(x) = c_2 + \alpha x + \int_0^x \left(\int_0^y f(s)ds\right) dy. \tag{26}$$

Clearly, the solution (26) satisfies (22) for all $c_2 \in \mathbb{R}$, provided (25) holds. On the other hand, if $f(x)$ does not satisfy (25) then the boundary value problem (22) has no solution.

Example: Let us provide another example of a linear BVP with no solution. To this end, consider

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 0, \qquad y(\pi) = 1. \tag{27}$$

The flow generated by the corresponding first-order system

$$\frac{dv_1}{dt} = v_2, \qquad \frac{dv_2}{dt} = -v_1 \tag{28}$$

is a center. There are in principle infinite trajectories that start from $v_1(0) = y(0) = 0$ and end at 1. None of them though makes the trip exactly in $t = \pi$ time units. Indeed the general solution of the ODE $\ddot{y} + y = 0$ is (27) is

$$y(t) = c_1 \cos(t) + c_2 \sin(t). \tag{29}$$

Setting $y(0) = 0$ yields $c_1 = 0$. Setting $y(\pi) = 1$ yields $c_1 = 1$, which is incompatible with $c_1 = 0$.

Ill-posed nonlinear boundary value problems

Next, consider the following nonlinear boundary value problem for a second-order ODE with Dirichlet conditions

$$\begin{cases} \dfrac{d^2y}{dt^2} = f\left(\dfrac{dy}{dt}, y, t\right) & t \in [0, T] \\ y(0) = \alpha \\ y(T) = \beta \end{cases} \tag{30}$$

It is easy to show by a simple example that this problem can have an infinite number of solutions or no solution at all. To this end, consider the pendulum equations

$$\begin{cases} \dfrac{d^2\theta}{dt^2} = -\sin(\theta) \\ \theta(0) = \dfrac{\pi}{2} \\ \theta(T) = 0 \end{cases} \tag{31}$$

where T is the time that it takes to the pendulum to reach the vertical position after swinging from right to left only once from a zero velocity initial condition. It is clear that there are multiple solutions to this problem. In Figure 3 we sketch two of such initial velocities, and corresponding trajectories. If we change the boundary condition $\theta(0)$ to $\theta(0) = \pi$ then of course there are no solutions to (32). In fact, the pendulum wont move from the vertical (unstable) position, and therefore there is no way it can reach the point $\theta(T) = 0$.

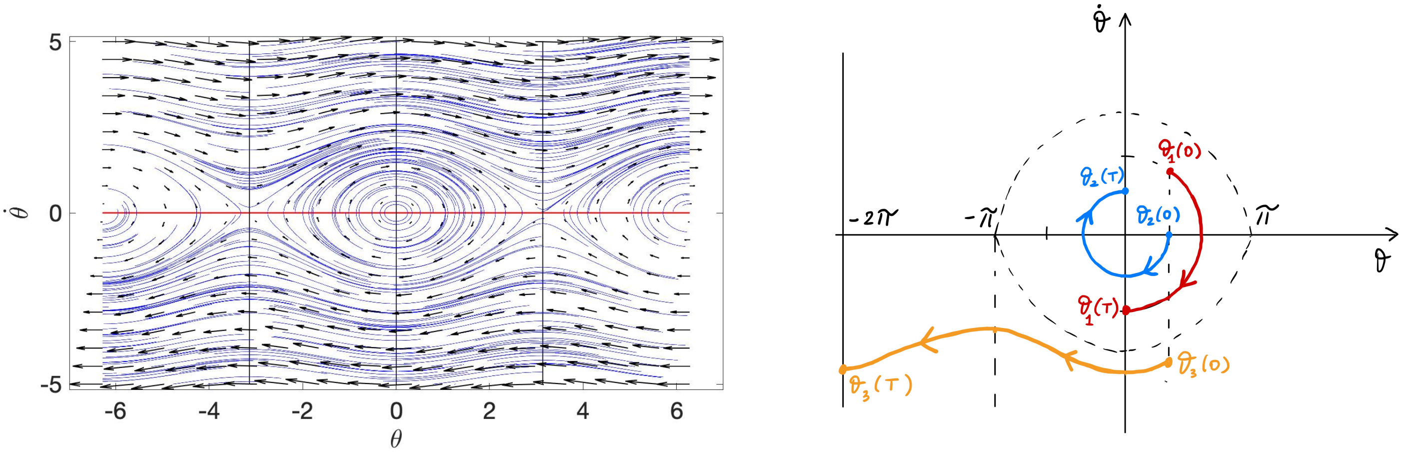

Figure 3: Phase portrait of the pendulum equation $\ddot{\theta} = -\sin(\theta)$ and sketch of two solutions (ϑ_1 and ϑ_2) of the BVP (32). The third solution technically does not satisfy $\theta(T) = 0$ but rather $\theta(T) = -2\pi$ which is physically equivalent, but mathematically different.

Upon definition of $y_1 = \theta$ and $y_2 = d\theta/d$ we can write the boundary value problem (32) in the form (19) as

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -\sin(y_1) \\ y_1(0) = \dfrac{\pi}{2} \\ y_1(T) = 0 \end{cases} \tag{32}$$

In this case, the $\boldsymbol{g} = (g_1, g_2)$ that defines the boundary conditions has components

$$g_1(\boldsymbol{y}(0), \boldsymbol{y}(T)) = y_1(0) - \frac{\pi}{2}, \qquad g_2(\boldsymbol{y}(0), \boldsymbol{y}(T)) = y_1(T). \tag{33}$$

Appendix A: Proof of Theorem 1

Upon definition of $\boldsymbol{z} = d\boldsymbol{y}/dt$ we can write (20) as

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \end{bmatrix}. \tag{34}$$

Suppose that $\boldsymbol{A}$ is diagonalizable, and let $\boldsymbol{P}$ and $\boldsymbol{\Lambda}$ be the matrix of eigenvectors and the diagonal matrix of eigenvalues of $\boldsymbol{A}$, i.e.,

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}. \tag{35}$$

Consider the transformation induced by the invertible block matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix}. \tag{36}$$

Clearly,

$$\underbrace{\begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix}}_{\boldsymbol{H}} \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{\Lambda} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} \boldsymbol{P}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}^{-1} \end{bmatrix}}_{\boldsymbol{H}^{-1}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{37}$$

By applying $\boldsymbol{H}^{-1}$ to the system (34) we obtain

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{bmatrix} = \underbrace{\begin{bmatrix}\boldsymbol{0} & \boldsymbol{\Lambda}\\ \boldsymbol{I} & \boldsymbol{0}\end{bmatrix}}_{\boldsymbol{C}}\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{bmatrix}, \qquad \text{where} \qquad \begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{bmatrix} = \begin{bmatrix}\boldsymbol{P}^{-1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{P}^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}\\ \boldsymbol{y}\end{bmatrix}. \tag{38}$$

The solution to this ODE (treated as initial value problem with *unknown* $\boldsymbol{q}(0)$) is

$$\begin{bmatrix}\boldsymbol{q}(t)\\ \boldsymbol{p}(t)\end{bmatrix} = e^{t\boldsymbol{C}}\begin{bmatrix}\boldsymbol{q}(0)\\ \boldsymbol{P}^{-1}\boldsymbol{y}(0)\end{bmatrix}. \tag{39}$$

Setting the boundary conditions $\boldsymbol{y}(0) = \boldsymbol{\alpha}$ and $\boldsymbol{y}(T) = \boldsymbol{\beta}$ yields

$$\begin{bmatrix}\boldsymbol{q}(T)\\ \boldsymbol{P}^{-1}\boldsymbol{\beta}\end{bmatrix} = e^{\boldsymbol{C}}\begin{bmatrix}\boldsymbol{q}(0)\\ \boldsymbol{P}^{-1}\boldsymbol{\alpha}\end{bmatrix}. \tag{40}$$

The exponential matrix $e^{\boldsymbol{C}}$ has the following structure

$$e^{\boldsymbol{C}} = \begin{bmatrix}\boldsymbol{D}_1 & \boldsymbol{D}_2\\ \boldsymbol{D}_3 & \boldsymbol{D}_1\end{bmatrix}, \tag{41}$$

where $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$ are all diagonal matrices. Moreover, $\boldsymbol{D}_1$ and $\boldsymbol{D}_3$ are *invertible*. Substituting (41) into (40) gives

$$\begin{bmatrix}\boldsymbol{q}(T)\\ \boldsymbol{P}^{-1}\boldsymbol{\beta}\end{bmatrix} = \begin{bmatrix}\boldsymbol{D}_1 & \boldsymbol{D}_2\\ \boldsymbol{D}_3 & \boldsymbol{D}_1\end{bmatrix}\begin{bmatrix}\boldsymbol{q}(0)\\ \boldsymbol{P}^{-1}\boldsymbol{\alpha}\end{bmatrix}. \tag{42}$$

This equation allows us to determine $\boldsymbol{q}(0)$ *uniquely* for any given $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ In fact, the second equation in (42) can be written as

$$\boldsymbol{D}_3\boldsymbol{q}(0) = \boldsymbol{P}^{-1}\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{P}^{-1}\boldsymbol{\alpha} \qquad \Leftrightarrow \qquad \boldsymbol{q}(0) = \boldsymbol{D}_3^{-1}\boldsymbol{P}^{-1}\left(\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{\alpha}\right). \tag{43}$$

Hence, we proved that for every given $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ there exists *unique* initial "velocity"

$$\boldsymbol{z}(0) = \boldsymbol{P}\boldsymbol{q}(0) = \boldsymbol{P}\boldsymbol{D}_3^{-1}\boldsymbol{P}^{-1}\left(\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{\alpha}\right) \tag{44}$$

such that the system (34) integrated forward in time from the initial condition

$$\begin{bmatrix}\boldsymbol{z}(0)\\ \boldsymbol{y}(0)\end{bmatrix} = \begin{bmatrix}\boldsymbol{P}\boldsymbol{D}_3^{-1}\boldsymbol{P}^{-1}\left(\boldsymbol{\beta} - \boldsymbol{D}_1\boldsymbol{\alpha}\right)\\ \boldsymbol{\alpha}\end{bmatrix} \tag{45}$$

yields the solution to $\boldsymbol{y}(T) = \boldsymbol{\beta}$ at time T. By leveraging the existence and uniqueness of solutions to the initial value problem (39) we conclude that the two-point boundary value problem for the ODE (20) with Dirichlet boundary conditions has a unique solution.

If we drop the assumption of diagonalizability of $\boldsymbol{A}$ and replace the diagonal matrix $\boldsymbol{\Lambda}$ with its block diagonal Jordan form $\boldsymbol{J}$, then the ODE (20) with Dirichlet boundary conditions still has a unique solution. In fact, in this case the matrix exponential $e^{\boldsymbol{C}}$ is still a block matrix in the form (41), but with upper triangular $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ and $\boldsymbol{D}_3$. Moreover, $\boldsymbol{D}_1$ and $\boldsymbol{D}_3$ are invertible. Hence (43) still holds.

References

[1] D. W. Hahn and M. N. Özisik. *Heat Conduction*. Wiley, third edition, 2012.