One-dimensional dynamical systems

Consider the following initial value problem for one ODE[1]

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $f : D \mapsto \mathbb{R}$, $D \subseteq \mathbb{R}$ is a subset of $\mathbb{R}$, $x(t) \in D$ for each $t \in [0, T[$, and $x_0 \in D$. We know that for the initial value problem (1) to be well-posed – that is, for a unique solution to exist – it is both necessary and sufficient that $f(x)$ be *Lipschitz continuous* in the domain D.

Definition 1 (Lipschitz continuity)**.** Let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. We say that $f : D \to \mathbb{R}$ is Lipschitz continuous in D if there exists a positive constant $0 \leq L < \infty$ (Lipschitz constant) such that

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2| \qquad \text{for all } x_1, x_2 \in D. \tag{2}$$

The smallest number L^* such that the inequality above is satisfied is called "best" Lipschitz constant.

Lipschitz continuity is stronger than continuity, which requires only that[2]

$$\lim_{y_1 \to y_2^{\pm}} |f(y_1) - f(y_2)| = 0 \quad \text{for all } y_2 \in D. \tag{3}$$

In fact, Lipschitz continuity implies that the rate at which $f(x_1)$ approaches $f(x_2)$ as $x_1 \to x_2$ cannot be larger than L. In other words, a Lipschitz continuous function $f(x)$ has a growth rate that is bounded by L for all x in D.

Example: Let $D = [-1, 1]$ be a closed interval, i.e., an interval including the endpoints -1 and 1. The function $f(x) = x^{1/3}$ is continuous in D for all $t \in \mathbb{R}$ (see Figure 1). However, $f(x)$ is not Lipschitz continuous in D. The problem here is that $f(x)$ has infinite "slope" at the point $x = 0$. In other words, there is no constant $0 \leq L < \infty$ such that

$$|f(x) - f(0)| \leq L\,|x - 0| \quad \text{for all} \quad x \in D. \tag{4}$$

This can be seen by substituting $f(x) = y^{1/3}$ in the Lipschitz continuity condition (4)

$$|f(x)| \leq L\,|x| \quad \Rightarrow \quad \left|\frac{x^{1/3}}{x}\right| = \left|\frac{1}{x^{2/3}}\right| \leq L \quad \text{for all} \quad x \in D. \tag{5}$$

Clearly, if we send x to zero we have that $\left|x^{-2/3}\right| \to \infty$, which cannot be bounded from above by any finite constant L. In other words, $f(x)$ is not Lipschitz continuous in D because its growth rate at $x = 0$ is too large. However, if we remove $x = 0$ and consider, e.g., the domain

$$D = \left[\frac{1}{10}, 1\right] \tag{6}$$

then $f(x)$ is Lipschitz continuous in D. Actually more than just continuous, $f(x)$ is infinitely-differentiable with continuous derivatives in D. Finally we notice that $f(x)$ is not Lipschitz continuous in the open interval $D =]0, 1]$. In fact the growth rate of $f(x)$ cannot be bounded by a finite constant L as $x \to 0^+$.

[1] The ODE (1) is called "autonomous" if the right hand side f does not depend explicitly on t.

[2] The notation $x_1 \to x_2^{\pm}$ means that x_1 is approaching x_2 either from the left ("$-$") or from the right ("$+$"). Note that we can equivalently write (3) as

$$\lim_{x_1 \to x_2^{+}} f(x_1) = \lim_{x_1 \to x_2^{-}} f(x_1) = f(y_2).$$

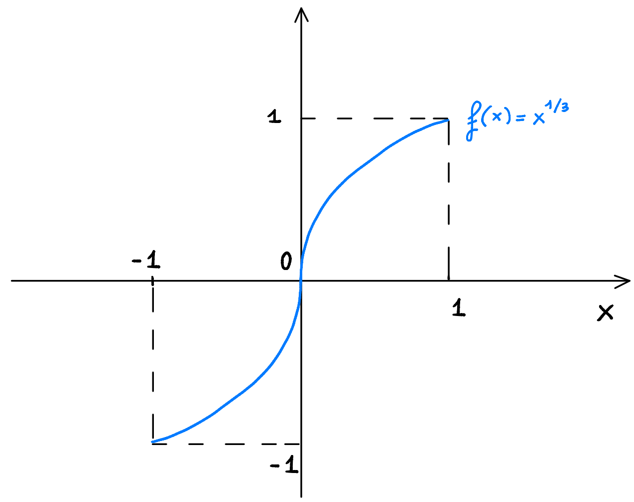


Figure 1: Sketch of the function $f(x) = x^{1/3}$ in $D = [-1, 1]$. The function is continuous in D, but it has an "infinite slope" at $x = 0$ and therefore it is not Lipschitz continuous in D.

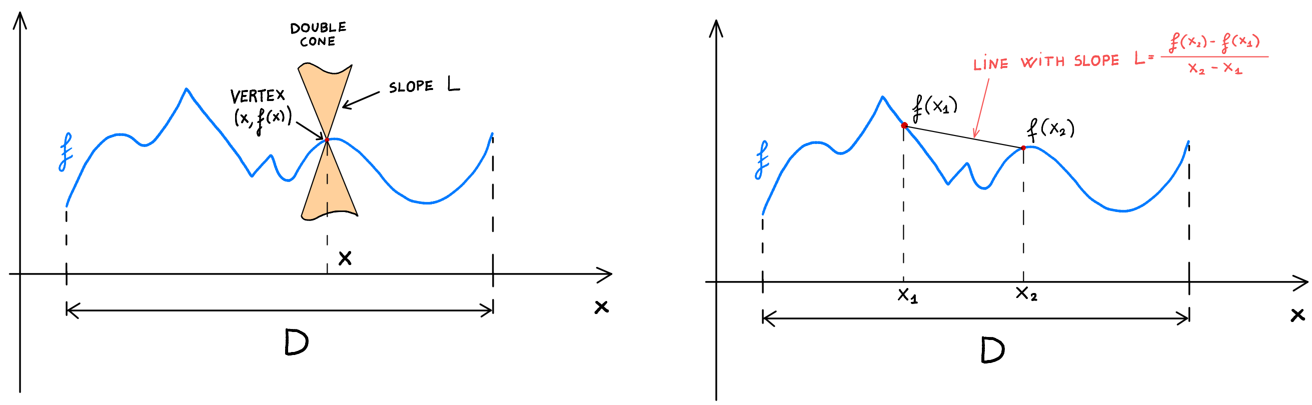


Figure 2: Geometric meaning of Lipschitz continuity.

Geometric meaning of the Lipschitz continuity condition. The Lipschitz continuity condition (2) has a nice geometric interpretation. In practice it says that the function $f(x)$ cannot enter a double cone with slope L and vertex placed on any point of the graph $(x, f(x))$ with $x \in D$. In other words, if we can slide the vertex of the double cone over the graph of the function $f(x)$ for all $x \in D$ and the function never enters the cone then $f(x)$ is Lipschitz continuous in D. To explain this, let us divide the inequality (2) by $|x_1 - x_2|$ (for $x_1 \neq x_2$). This yields

$$\underbrace{\left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right|}_{|K|} \leq L \quad \text{for all} \quad x_1, x_2 \in D. \tag{7}$$

As shown in Figure 2, K represents the slope of the line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. The "best" Lipshitz constant is obtained as

$$L^* = \max_{x_1, x_2 \in D} \left|\frac{f(x_1) - f(x_2)}{x_1 - x_2}\right|. \tag{8}$$

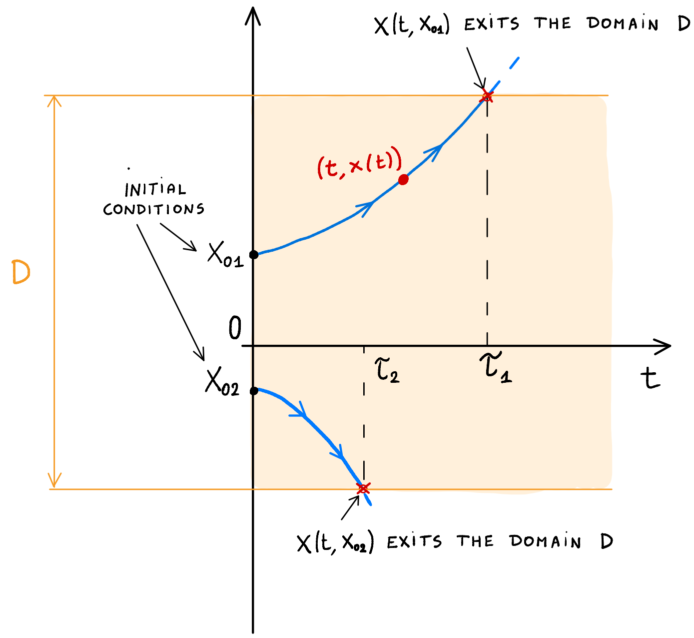


Figure 3: Geometric meaning of the existence and uniqueness theorem for the solution of one ODE.

Any finite number $L \geq L^*$ is still a Lipschitz constant, though not the best one. If the function $f(x)$ is continuously differentiable on a closed set $D \subset \mathbb{R}$ then

$$L^* = \max_{x \in D} \left| \frac{df(x)}{dx} \right| < \infty. \tag{9}$$

Lemma 1. If $f(x)$ is continuously differentiable on a closed set $D \subseteq \mathbb{R}$ then $f(x)$ is Lipschitz continuous in D.

Proof. By assumption the derivative of $df(x)/dx$ is continuous in the closed set $D \subseteq \mathbb{R}$. This implies that the minimum and the maximum of $df(x)/dx$ are attained at some points in D (Extreme Value Theorem). By using the mean value theorem we have that for any given x_1 and x_2

$$|f(x_1) - f(x_2)| = \left| \frac{df(x^*)}{dx} \right| |x_1 - x_2| \,. \tag{10}$$

where x^* is some point within the interval $[x_1, x_2] \subset D$. The point x^* depends on f, x_1 and x_2. The right hand side of (10) can be bounded as

$$|f(x_1) - f(x_2)| \leq \underbrace{\max_{x \in D} \left| \frac{df(x)}{dx} \right|}_{L^*} |x_1 - x_2| \qquad \text{for all} \quad x_1, x_2 \in D. \tag{11}$$

□

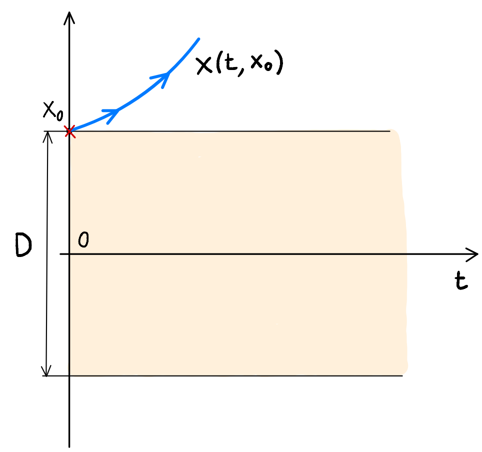
Example: The function $f(x) = x^2$ is of class C^∞ (infinitely differentiable with continuous derivatives) in any closed set $D \subset \mathbb{R}$. However, the function $f(x) = x^2$ is not Lipschitz continuous at $x = \pm\infty$, since the slope of the first-order derivative $f'(x) = 2x$ grows unboundedly as $x \to \pm\infty$.

Well-posedness of the Cauchy problem.

Next, we formulate the existence and uniqueness theorem for the solution of the first-order ODE (1).

Theorem 1. Let $D \subset \mathbb{R}$ be an open set, $x_0 \in D$. If $f : D \to \mathbb{R}$ is Lipschitz continuous in D then there exists a unique solution to the initial value problem (1) within the time interval $[0, \tau[$, where τ is the instant at which $x(t)$ exits[3] the domain D (see Figure 3). The solution $x(t)$ is continuously differentiable in $[0, \tau[$.

Remark: In Theorem 1, we required that D is an open set so that we can have solutions in D at least for some $t \in [0, \tau[$. On the other hand, if D is closed then we can pick x_0 right at the boundary of D so that the solution[4] $x(t) = X(t, x_0)$ never enters D, which is the region in which f is assumed to be Lipschitz continuous. In this case, the "exit time" τ may be zero, and Theorem 1 does not provide any information on the existence and uniqueness of the solution.



Global solutions. If $f(x)$ is Lipschitz continuous on the entire real line $\mathbb{R}$ then the solution to the initial value problem (1) is *global*. This means that the solution exists and is unique for all $t \geq 0$. In fact, $x(t)$ never exits the domain in which $f(x)$ is Lipschitz continuous, and therefore we can extend τ in Theorem 1 to infinity. It is important to emphasize that existence and uniqueness of the solution to (1) has nothing to do with the smoothness of $f(x)$ but rather with the rate at which $f(x)$ grows or decays.

Computing the solution of one-dimensional autonomous ODEs. The initial value problem (1) is separable, and it can be equivalently written in an integral form as

$$\int_{x_0}^{x(t)} \frac{dx}{f(x)} = t. \tag{12}$$

Hence, if we know how to compute the primitive of $1/f(x)$, i.e., the integral at the left hand side of (12), then we have an algebraic equation that relates $x(t)$, x_0 and t. This does not mean that we can always easily write $x(t)$ explicitly in terms of x_0 and t. This is demonstrated in the following simple example.

Example: Consider the initial value problem (1) and set

$$f(x) = \frac{1}{x^4 - x^2 + 1} \quad \text{and} \quad x_0 = 0. \tag{13}$$

[3]As shown in Figure 3, the "exit time" τ depends on D $f(x)$ and x_0.

[4]The nonlinear map $X(t, x_0)$ represents the solution of (1) corresponding to the initial condition x_0, where x_0 is left unspecified. As we shall see hereafter $X(t, x_0)$ is called *flow* generated by the dynamical system (1).

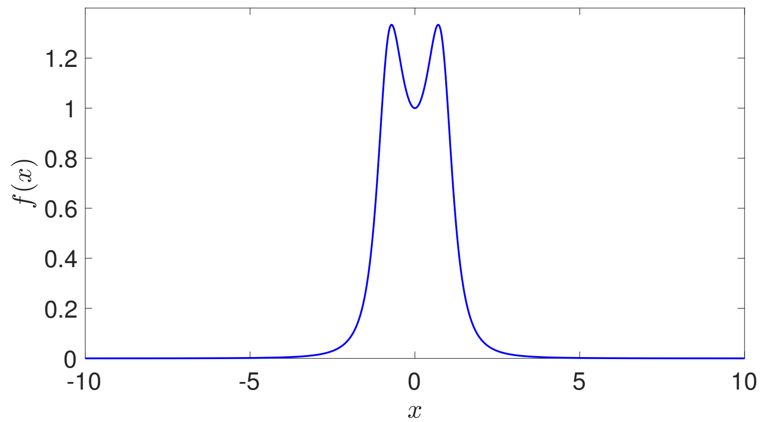


Figure 4: Plot of the function defined in equation (13).

As it is seen in Figure 4, $f(x)$ continuously differentiable in $\mathbb{R}$ with bounded derivative. Therefore, the solution of the initial value problem (1), with f and x_0 as in (13) is global, meaning that it exists and it is unique for all $t \geq 0$. A substitution of (13) into the integral equation (12) yields

$$\frac{x(t)^5}{5} - \frac{x(t)^3}{3} + x(t) = t. \tag{14}$$

Hence, to express $x(t)$ as a function of t we need to compute the roots of the fifth-order polynomial (14) as a function of t and among them select the one that passes through $x(0) = 0$.

Example: Consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = \sin(x) \\ x(0) = x_0 \end{cases} \tag{15}$$

where x_0 is any number in the interval $D = [0, \pi]$. The solution to (15) can be obtained by computing the integral[5]

$$\int_{x_0}^{x(t)} \frac{dx}{\sin(x)} = t \qquad \Rightarrow \qquad \left[\log\left(\left|\tan\left(\frac{x}{2}\right)\right|\right)\right]_{x_0}^{x(t)} = t \tag{16}$$

By using the properties of the logarithm we obtain

$$\log\left|\frac{\tan\left(\dfrac{x(t)}{2}\right)}{\tan\left(\dfrac{x_0}{2}\right)}\right| = t \qquad \Rightarrow \qquad x(t) = 2\arctan\left(e^t \tan\left(\frac{x_0}{2}\right)\right). \tag{17}$$

Note that

$$\lim_{t\to\infty} x(t) = \pi \tag{18}$$

The trajectories of the system (15) are shown in Figure 8.

[5] Recall that the primitive of $1/\sin(x)$ is

$$\log\left(\left|\tan\left(\frac{x}{2}\right)\right|\right).$$

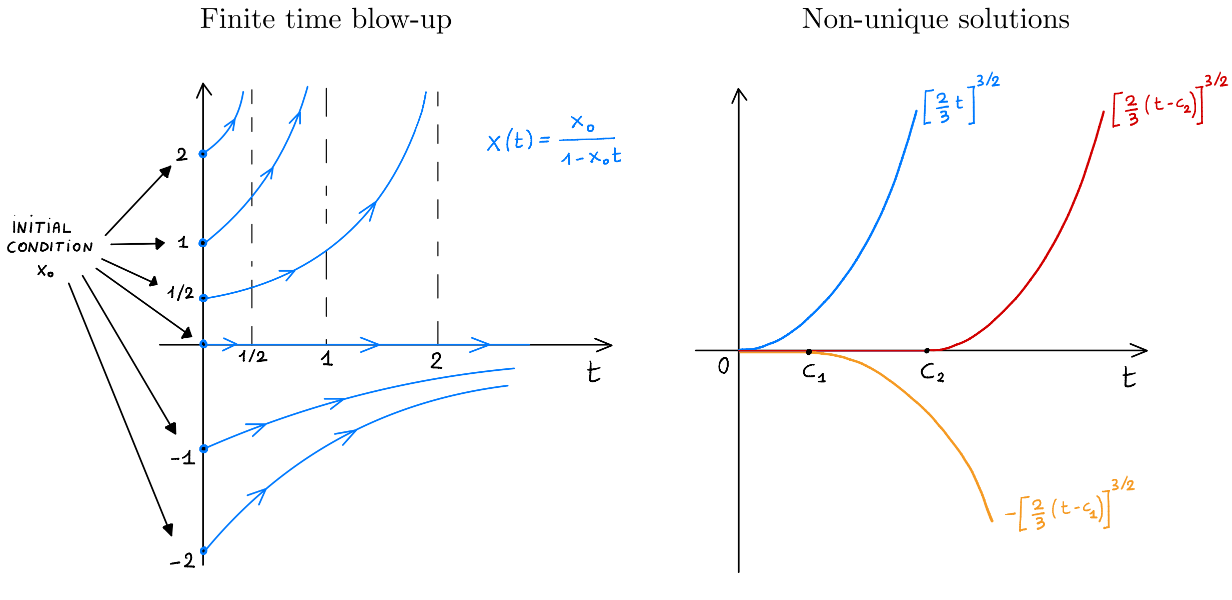


Figure 5: (left) Solutions of the initial value problem (19) for different initial conditions x_0. It is seen that for $x_0 > 0$ the solution blows up at the fine time $t^* = 1/x_0$. On the other hand, if $x_0 \leq 0$ the solution exists and is unique for all $t \geq 0$. (b) Solutions of the initial value problem (21) corresponding to the initial condition $x_0 = 0$. This problem has an infinite number of solutions.

Finite-time blow up and non-unique solutions. Hereafter we provide an example of an initial value problem the solution of which blows-up in a finite time, and an example of an initial value problem that has an infinite number of solutions.

- **Finite-time blow-up**: Consider the initial value problem

$$\frac{dx}{dt} = x^2 \qquad x(0) = x_0. \tag{19}$$

We know that $f(x) = x^2$ is not Lipschitz continuous at infinity. By using separation of variables, i.e., equation (12), it is straightforward to show that

$$\int_{x_0}^{x(t)} \frac{dx}{x^2} = -\frac{1}{x(t)} + \frac{1}{x_0} = t \quad \Rightarrow \quad x(t) = \frac{x_0}{1 - x_0 t}. \tag{20}$$

The function $x(t)$ clearly blows up to infinity as t approaches $1/x_0$ (from the left) for positive initial conditions x_0. On the other hand, if $x_0 \leq 0$ the solution exists and is unique for all $t \geq 0$.

- **Non-unique solutions:** Consider the initial value problem

$$\frac{dx}{dt} = x^{1/3} \qquad x(0) = 0. \tag{21}$$

We have seen that $f(x) = x^{1/3}$ is not Lipschitz continuous in any domain D that includes the point the point $x = 0$. Note that we are setting the initial condition exactly at the point in which the slope of $f(x)$ is infinity (see Figure 1). By using separation of variables it straightforward to show that a solution to (21) is

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}. \tag{22}$$

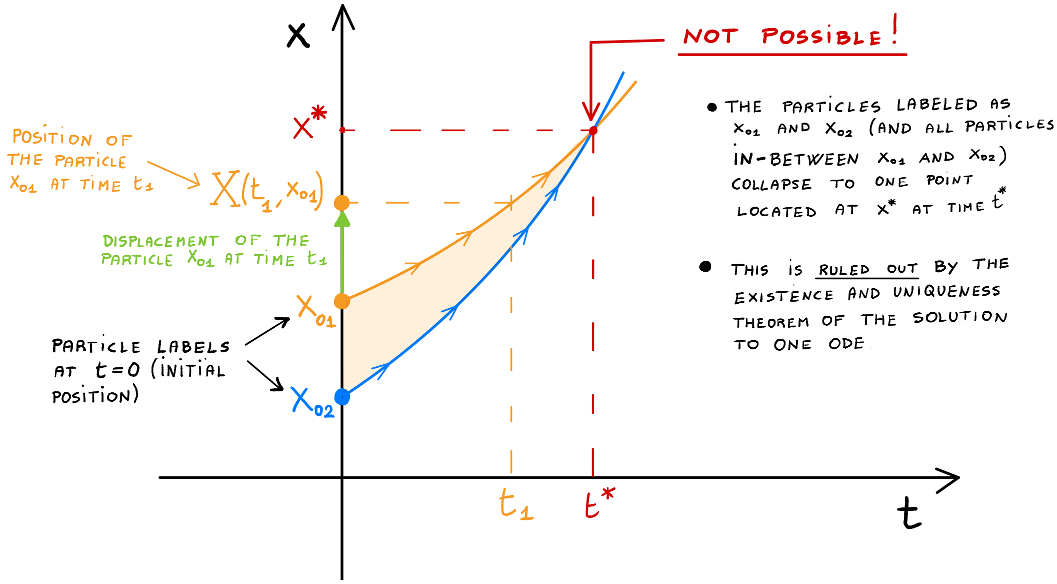


Figure 6: Trajectories corresponding to different initial conditions cannot intersect.

However, note that the functions

$$x(t) = \begin{cases} 0 & \text{for } 0 \le t < c \\ \pm\left(\dfrac{2}{3}(t-c)\right)^{3/2} & \text{for } t \ge c \end{cases} \tag{23}$$

are also solutions to (21) for all $c \geq 0$.

One-dimensional flows

We have seen that the initial value problem (1) admits a unique solution $x(t)$ (continuously differentiable in t) if $f(x)$ is Lipschitz continuous on an open subset $D \subset \mathbb{R}$ (Theorem 1), and if x_0 is chosen in D. This means that the solution $x(t)$ depends on $f(x)$ and x_0. We will denote the dependence of $x(t)$ on x_0 as $X(t, x_0)$, i.e.,

$$x(t) = X(t, x_0). \tag{24}$$

Let us first notice that because of the existence and uniqueness Theorem 1, it is not possible for two solutions corresponding to two different initial conditions to intersect at any finite time t (see Figure 6). This implies that $X(t, x_0)$ is invertible at each finite time[6] (see below), i.e., we can always identify which "particle" x_0 sits at location $x(t) = X(t, x_0)$ at time t. Moreover, it is impossible for two "particles" x_{01} and x_{02} to collide at any finite time, or for one particle to split into two or more particles (Figure 6). Next, we characterize how the flow $X(t, x_0)$ depends on the initial condition x_0 at each fixed time t.

Theorem 2 (Regularity of the ODE solution with respect to x_0)**.** Let $D \subset \mathbb{R}$ be an open set, $x_0 \in D$. If $f : D \to \mathbb{R}$ is Lipschitz continuous in D then the solution of the initial value problem (1), i.e., $X(t, x_0)$ (i.e., the flow generated by the ODE) is continuous in x_0. Moreover, if $f(x)$ is of class C^k in D (continuously differentiable k-times in D with continuous derivative), then $X(t, x_0)$ is of class C^k in D.

[6] Solutions corresponding to different initial conditions can, however, intersect at $t = \infty$, e.g., when there exist an attracting set such as a stable equilibrium point (proof below).

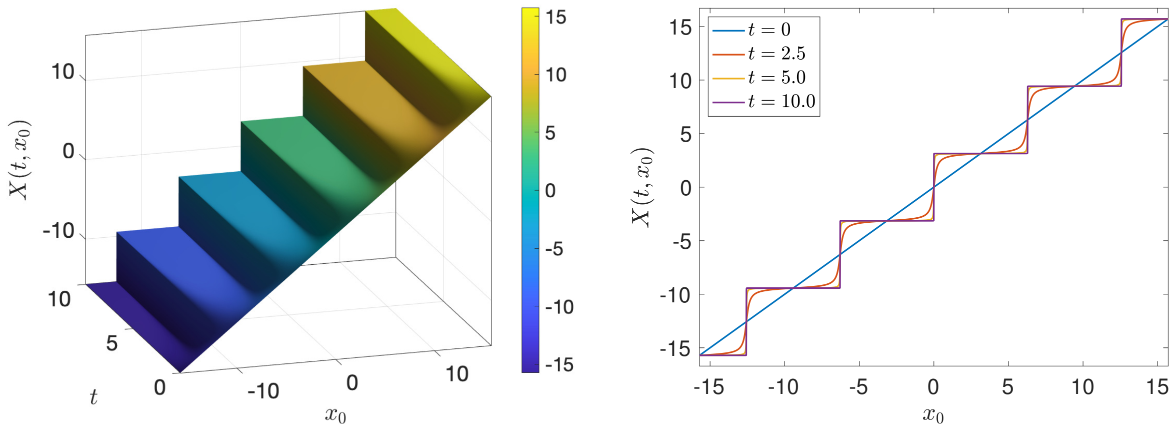


Figure 7: Visualization of the flow generated by the ODE (28).

The proof of this Theorem is provided in Appendix A. In summary, Theorem 2 states that the smoother $f(x)$, the smoother the dependency of $X(t, x_0)$ on x_0. The two-dimensional function $X(t, x_0)$ is called *flow generated by the dynamical system* (1), and it represents the full set of solutions to (1) for every initial condition x_0.

Theorem 3 (Regularity of the ODE solution with respect to t)**.** Let $D \subset \mathbb{R}$ be an open set, $x_0 \in D$. if $f(x)$ is of class C^k in D (continuously differentiable k-times in D with continuous derivative), then $X(t, x_0)$ is of class C^{k+1} in time.

Proof. The continuity of higher-order derivatives, and its link to the the regularity of $f(x)$ can be established by differentiating the ODE

$$\frac{dX(t, x_0)}{dt} = f(X(t, x_0)) \tag{25}$$

with respect to time. For instance, we have

$$\frac{d^2X(t, x_0)}{dt^2} = f'(X(t, x_0))f(X(t, x_0)), \tag{26}$$

$$\frac{d^3X(t, x_0)}{dt^3} = f''(X(t, x_0))f^2(X(t, x_0)) + [f'(X(t, x_0))]^2 f(X(t, x_0)). \tag{27}$$

At this point we can use the existence and uniqueness theorem for the solution of higher-dimensional dynamical systems to conclude that the derivatives $d^nX(t, x_0)/dt^n$ are continuous if $d^{n-1}f(x)/dx^{n-1}$ is continuous.

□

Example: In Figure 7 we visualize the flow generated by the ODE

$$\frac{dx}{dt} = \sin(x) \tag{28}$$

for all $x_0 \in [-5\pi, 5\pi]$ and $t \in [0, 10]$. Such flow was computed by solving the ODE (28) numerically for a large number of initial conditions x_0. By using on Theorem 2 and Theorem 3 we conclude that the flow generated by (28) is of class C^∞ in both t and x_0.

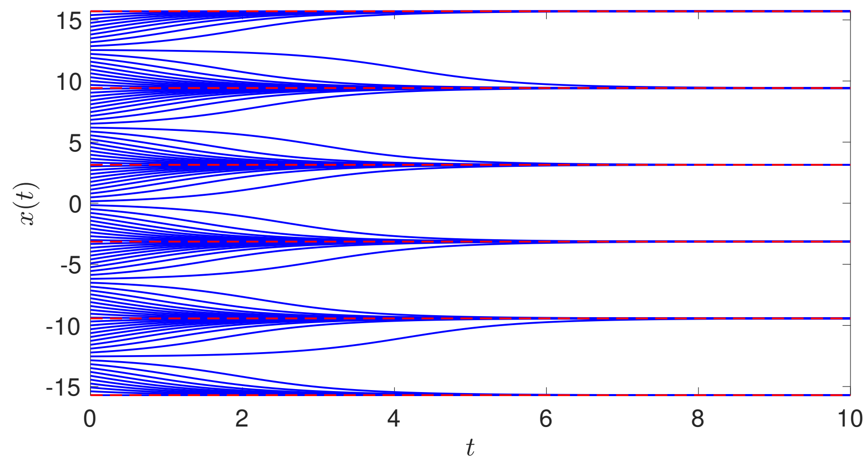


Figure 8: Trajectories of the dynamical system (28) corresponding to 100 evenly spaced initial conditions in $[-5\pi, 5\pi]$. All trajectories are computed numerically. The red dashed lines represent the stable fixed points (equilibria) of the system.

Similarly, in Figure 8 we plot the trajectories of the system (28) corresponding to an evenly-spaced grid of 100 initial conditions in $[-5\pi, 5\pi]$.

Properties of the flow. The flow generated by one dimensional dynamical systems of the form (1) satisfies the following properties:

- $X(0, x_0) = x_0$. This means that at $t = 0$ the mapping $X(t, x_0)$ is the identity.
- $X(t, x_0)$ is monotonic in x_0 for each fixed t, i.e.,

$$X(t, x_{02}) > X(t, x_{01}) \quad \text{for all } x_{02} > x_{01}. \tag{29}$$

Assuming $f \in C^1(D)$ This property can be proved by substituting $x(t) = X(t, x_0)$ into the ODE $dx/dt = f(x)$ and differentiating it with respect to x_0. This yields

$$\frac{\partial}{\partial x_0}\left(\frac{dX(t,x_0)}{dt}\right) = \frac{\partial f(X(t,x_0))}{\partial x_0} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial X(t,x_0)}{\partial x_0}\right) = f'(X(t,x_0))\frac{\partial X(t,x_0)}{\partial x_0}. \tag{30}$$

This ODE is linear and can be easily integrated in time from the initial condition

$$\frac{\partial X(0,x_0)}{\partial x_0} = 1 \tag{31}$$

to obtain[7]

$$\frac{\partial X(t,x_0)}{\partial x_0} = \exp\left[\int_0^t f'(X(\tau,x_0))d\tau\right]. \tag{34}$$

[7]Equation (34) characterizes the dynamics of an infinitesimal "line element" with length dx_0 as it is "transported" by the flow $X(t, x_0)$. In fact, from (34) it follows that

$$dX(t,x_0) = dx_0 \exp\left[\int_0^t f'(X(\tau,x_0))d\tau\right]. \tag{32}$$

Moreover, if x_0 is a fixed-point, i.e. if $X(\tau, x_0) = x_0$, then

$$dX(t,x_0) = dx_0 e^{tf'(x_0)}. \tag{33}$$

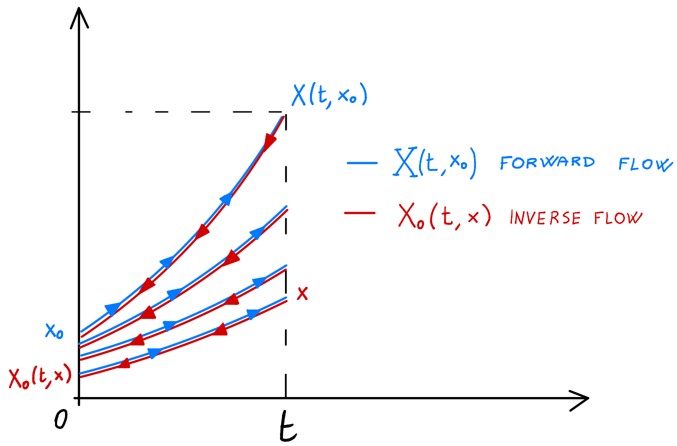


Figure 9: Illustration of forward and inverse flows.

The right hand side of (34) is strictly positive for each $t \geq 0$, which implies

$$\frac{\partial X(t, x_0)}{\partial x_0} > 0 \qquad \text{for each finite } t \geq 0. \tag{35}$$

This proves that if $f \in C^1(D)$ then[8] the flow map $X(t, x_0)$ is monotonic in x_0 and therefore invertible for each finite t.

- $X(t, x_0)$ satisfies the composition rule $X(t+s, x_0) = X(t, X(s, x_0)) = X(s, x(t, x_0))$. This property is called "semi-group property" of the flow and it follows from the fact that we can restart integration of the ODE (1) at time t (or time s) from the new initial condition $X(t, x_0)$ (or $X(s, x_0)$) to get to the final integration time $s + t$. Again, this property holds because of the existence and uniqueness Theorem 1.

Inverse flow. The monotonicity property (29) ensures that the flow is *invertible* for every finite $t \geq 0$ (provided the forward flow exists). In other words, for any location x, there exists a unique initial x_0 that gets to x at time t. As mentioned above, this implies that no two distinct initial conditions x_0 can occupy the same location x at the same time – i.e., the trajectories of the initial value problem (1) cannot intersect (see Figure 6). The invertibility of $X(t, x_0)$ with respect to x_0 for each fixed t allows us to define the *inverse flow*, which determines the label x_0 of the particle located at position x at time t. In practice, the inverse flow can be computed by integrating the initial value problem (1) backward in time from the

[8]If $f(x)$ is not differentiable but just Lipschitz continuous then we can prove the monotonicity statement using the Grönwall lemma. To this end, let $x_{02} > x_{01}$ and consider

$$\frac{d}{dt}\left(X(t, x_{02}) - X(t, x_{01})\right) = f(X(t, x_{02})) - f(X(t, x_{01})). \tag{36}$$

Since f is Lipschitz we have

$$-L(x_{02} - x_{01}) \leq f(X(t, x_{02})) - f(X(t, x_{01})) \leq L(x_{02} - x_{01}) \tag{37}$$

which implies that

$$\frac{d}{dt}\left(X(t, x_{01}) - X(t, x_{02})\right) \leq L(x_{02} - x_{01}). \tag{38}$$

By using Grönwall lemma we obtain

$$X(t, x_{01}) \leq X(t, x_{02}) + (x_{01} - x_{02})e^{L(x_{02} - x_{01})} < X(t, x_{02}). \tag{39}$$

Hence, if $x_{02} > x_{01}$ then $X(t, x_{02}) > X(t, x_{01})$

condition x at time t to $t=0$. This is equivalent to integrating forward in time the ODE system with a reversed velocity vector (see Figure 9).

$$\begin{cases} \dfrac{dx}{dt} = -f(x) \\ x(0) = x \end{cases} \tag{40}$$

The flow associated with this system will be denoted as $X_0(t,x)$. Clearly, for each fixed t the inverse flow $X_0(t,x)$ is the inverse of the forward flow $X(t,x_0)$, i.e.,

$$X(t, X_0(t,x)) = x, \qquad X_0(t, X(t,x_0)) = x_0. \tag{41}$$

Flow map equation. By using (41) is is easy to show that the flow $X(t,x_0)$ generated by the initial value problem (1) is governed by the first-order partial differential equation[9] (PDE)

$$\begin{cases} \dfrac{\partial X(t,x_0)}{\partial t} - f(x_0)\dfrac{\partial X(t,x_0)}{\partial x_0} = 0 \\ X(0,x_0) = x_0 \end{cases} \tag{42}$$

This can be verified, e.g., by substituting the flow

$$X(t,x_0) = \frac{x_0}{1-x_0 t} \tag{43}$$

generated by (19) into (42). Indeed, by computing the derivatives

$$\frac{\partial X(t,x_0)}{\partial t} = \frac{x_0^2}{(1-x_0t)^2}, \qquad \frac{\partial X(t,x_0)}{\partial x_0} = \frac{1}{(1-x_0t)^2}. \tag{44}$$

and recalling that $f(x_0) = x_0^2$ in this case, we see that (42) is identically satisfied. Equation (9) is a first-order hyperbolic PDE that can be solved with the method of characteristics, or numerically with finite differences or spectral methods, to obtain the flow map. The solution to (42) can be formally expressed in terms of an exponential operator known as *Koopman operator*. To this end, we first define the linear (differential) operator

$$K(x_0) = f(x_0)\frac{\partial}{\partial x_0}, \tag{45}$$

which is known as *generator* of the Koopman operator. This allows us to write (42) as

$$\frac{\partial X(t,x_0)}{\partial t} = K(x_0)X(t,x_0), \tag{46}$$

and therefore obtain the formal solution

$$X(t,x_0) = e^{tK(x_0)}x_0, \tag{47}$$

where $e^{tK(x_0)}$ is the Koopman operator. In this form, it is immediate to prove (at least formally) the semi-group property of the flow we discussed previously. In fact,

$$X(t+s,x_0) = e^{(t+s)K(x_0)}x_0 = e^{tK(x_0)}e^{sK(x_0)}x_0 = e^{tK(x_0)}X(s,x_0) = X(t,X(s,x_0)). \tag{48}$$

[9] The proof of (42) follows immediately by differentiating the identity $X(t,X_0(t,x)) = x$ in (41) with respect to t, and then using the inverse flow equation (40), i.e., $\partial X_0/\partial t = -f(X_0)$.

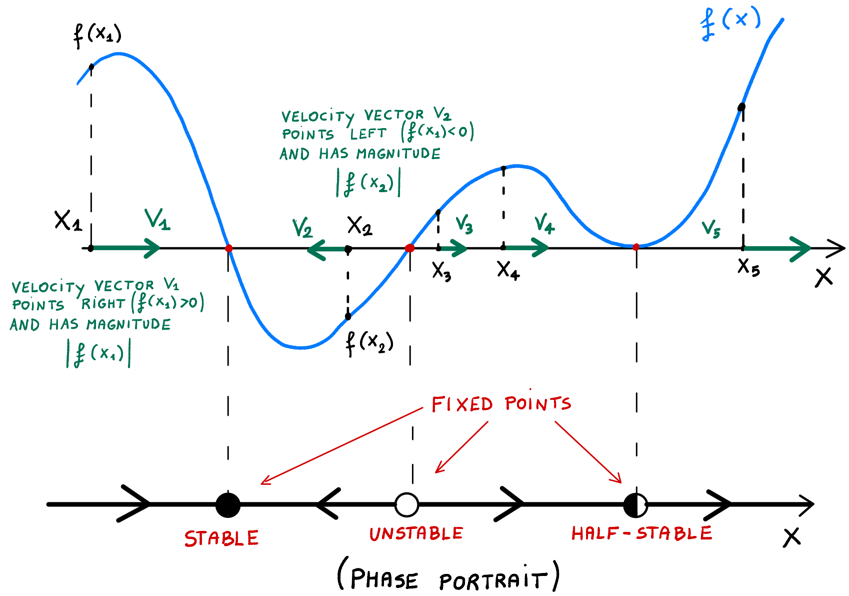


Figure 10: Vector field associated with $f(x)$, fixed points, and phase portrait.

Similarly, the inverse flow $X_0(t,x)$ defined by the dynamical system (40) is governed by the PDE[10]

$$\begin{cases} \dfrac{\partial X_0(t,x)}{\partial t} + f(x)\dfrac{\partial X_0(t,x)}{\partial x} = 0 \\ X_0(0,x) = x \end{cases} \tag{49}$$

The solution to this PDE is

$$X_0(t,x) = e^{-tK(x)}x. \tag{50}$$

Geometric approach to one-dimensional dynamical systems

Dynamical systems of the form (1) generate a flow $X(t,x_0)$ that maps each initial condition x_0 to the solution of the ODE at time t. If we interpret x_0 as the initial position of a particle moving along a line (i.e., the phase space), then from elementary mechanics we know that $dX(0,x_0)/dt = f(x_0)$ represents the velocity of that particle at time $t=0$. Given a function $f(x)$, we can immediately plot the *vector field*[11] associated with the dynamical system. This field visually indicates how fast a particle located at any position x moves to the left or right. Clearly, if the velocity vector $f(x)$ is equal to zero at some location x^*, then any particle placed at that point will not move as time evolves. These points are called *fixed points* (or *equilibria*) of the dynamical system (1). Fixed points can be rigorously defined as the values $x^* \in \mathbb{R}$ such that, for all $t \geq 0$,

$$X(t,x^*) = x^*. \tag{51}$$

By differentiating the previous equation with respect to time yields

$$0 = \frac{\partial X(t,x^*)}{\partial t} = f(X(t,x^*)) = f(x^*). \tag{52}$$

[10] Equation (49) follows by differentiating the identity $X_0(t,X(t,x_0)) = x_0$ in (41) with respect to t, and then using the forward flow equation $\partial X/\partial t = f(X)$.

[11] A vector field is a vector that is continuously indexed by one or more variables. For one-dimensional dynamical systems, the vector field $f(x)$ is indexed by the coordinate x.

Therefore, the fixed points of the system (1) are *zeros* of the nonlinear function $f(x)$, i.e.

$$f(x^*) = 0, \tag{53}$$

(see Figure 10). The calculation of the fixed points can be done analytically for prototype dynamical systems. In general, computing the fixed points requires a root-finding numerical algorithm such as the Newton's method.

Remark: The (Lipschitz) continuity condition on $f(x)$ in Theorem 1 imposes topological constraints on the distribution of fixed points. Specifically, fixed points facing each other cannot be both stable or unstable, but rather they must have opposite stability properties (Figure 10).

Stability analysis of fixed points.

A fixed point x^* is said to be *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ (dependent on ϵ) such that

$$|x_0 - x^*| < \delta \quad \Rightarrow \quad |X(t, x_0) - x^*| < \epsilon \qquad \forall t \geq 0. \tag{54}$$

A fixed point x^* is said to be *asymptotically stable* if

$$\lim_{t\to\infty} |X(t, x_0) - x^*| = 0, \tag{55}$$

for all x_0 in some neighborhood of x^*. In other words, stable fixed points attract trajectories of the dynamical system from both left and right (see Figure 10). Of course, by looking at the graph of $f(x)$ we can immediately infer the stability properties of all fixed points. This analysis can also be carried out analytically using *linearization*. The idea is straightforward: if $f(x)$ is smooth (at least continuously differentiable), then the more we "zoom in" near a fixed point x^*, the more $f(x)$ resembles a linear function[12] in a neighborhood of x^*. Thus, it can be approximated by the first-order term in its Taylor series expansion. In other words, by zooming in, we are effectively studying the *local dynamics* of the system near the fixed point. To this end, let us choose an initial condition x_0 that is sufficiently close to the fixed point x^*, say $x_0 - x^* = 10^{-10}$. By continuity, the flow $X(t, x_0)$ will map x_0 to a position that remains close to x^*, at least for some time (see Figure 11). The distance between $X(t, x_0)$ and the fixed point x^* can be expressed as a function[13]

$$\eta(t, x_0) = X(t, x_0) - x^* \qquad \Leftrightarrow \qquad X(t, x_0) = \eta(t, x_0) + x^*. \tag{57}$$

Substituting $X(t, x_0) = \eta(t, x_0) + x^*$ into (1) yields

$$\begin{cases} \dfrac{d\eta}{dt} = f(\eta + x^*) \\ \eta(0, x_0) = x_0 - x^* \end{cases} \tag{58}$$

If $\eta(0, x_0)$ is very small, then $\eta(t, x_0)$ remains small as well (at least for some time). This allows us to expand $f(\eta + x^*)$ in a Taylor series around x^*:

$$f(\eta(t, x_0) + x^*) = \underbrace{f(x^*)}_{=0} + f'(x^*)\eta(t, x_0) + \frac{1}{2} f''(x^*)\eta(t, x_0)^2 + \cdots \tag{59}$$

[12] This is not true for all functions. For example, if we zoom-in $f(x) = x^3$ at $x = 0$ we always find a cubic function!

[13] Note that

$$\eta(0, x_0) = X(0, x_0) - x^* = x_0 - x^*. \tag{56}$$

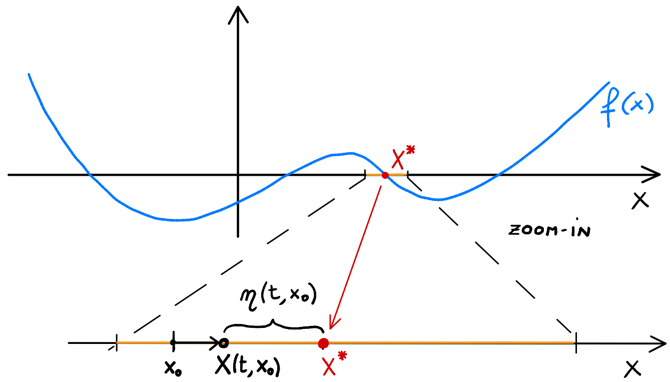


Figure 11: Linearization nearby the fixed point x^*.

Hence, to first order in η, we obtain the linear initial value problem

$$\begin{cases} \dfrac{d\eta}{dt} = f'(x^*)\,\eta \\ \eta(0, x_0) = x_0 - x^* \end{cases} \tag{60}$$

The solution of (60) is

$$\eta(t, x_0) = (x - x_0)e^{f'(x^*)t}. \tag{61}$$

The last equation allows us to conclude that

- $f'(x^*) < 0 \quad \Rightarrow \quad x^*$ is asymptotically stable
- $f'(x^*) > 0 \quad \Rightarrow \quad x^*$ is unstable
- $f'(x^*) = 0 \quad \Rightarrow \quad$ results of linear stability analysis are inconclusive.

If $f'(x_0) = 0$ then need to expand f to higher order in η, and *solve a nonlinear ODE* to classify the stability of the fixed point x^*.

Example: The dynamical system

$$\frac{dx}{dt} = \underbrace{x^2 - 1}_{f(x)} \tag{62}$$

has two fixed points located at $x^*_{1,2} = \pm 1$. Of course, $f'(x) = 2x$. By evaluating $f'(x)$ at the fixed points we see that $f'(1) = 2 > 0$ and $f'(-1) = -2 < 0$. Hence $x^*_1 = 1$ is unstable, and $x^*_2 = -1$ is asymptotically stable.

Example: The dynamical system

$$\frac{dx}{dt} = 1 + \sin(x) \tag{63}$$

has a global solution for all initial conditions x_0, and an infinite number of fixed points located at (see Figure 12)

$$x^*_k = \frac{3\pi}{2} + 2k\pi. \tag{64}$$

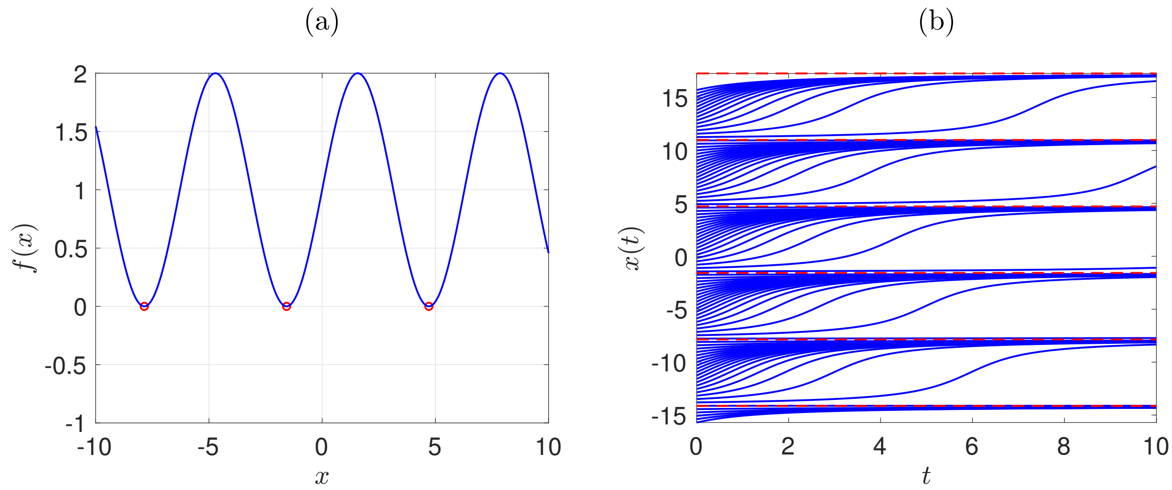


Figure 12: (a) Graph of function the $f(x) = 1 + \sin(x)$ and some of its fixed points (red circles). (b) Trajectories of the dynamical system (63).

By expanding $f(x) = 1 + \sin(x)$ in a Taylor series at $x_0^* = 3\pi/2$ we obtain

$$\begin{aligned}\sin\left(\eta + \frac{3\pi}{2}\right) &= \sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right)\eta - \frac{1}{2}\sin\left(\frac{3\pi}{2}\right)\eta^2 + \cdots \\ &= -1 + \frac{\eta^2}{2} + \cdots, \end{aligned} \tag{65}$$

i.e.,

$$1 + \sin\left(\eta + \frac{3\pi}{2}\right) = \frac{\eta^2}{2} + \cdots. \tag{66}$$

Note that in this case, the results of the linear stability analysis are inconclusive, as the coefficient multiplying the linear term in η, namely, $\cos(3\pi/2)$, is zero. Substituting (66) back into (58) yields the quadratic system

$$\begin{cases} \dfrac{d\eta}{dt} = \dfrac{\eta^2}{2} \\ \eta(0, x_0) = x_0 - \dfrac{3\pi}{2} \end{cases} \tag{67}$$

We computed the analytical solution to this system before (see Eq. (20)),

$$\eta(t, x_0) = \frac{\left(x_0 - \dfrac{3\pi}{2}\right)}{1 - \left(x_0 - \dfrac{3\pi}{2}\right)\dfrac{t}{2}}. \tag{68}$$

Clearly, if $x_0 > 3\pi/2$ the trajectory tends to move further away from the fixed point $x_0^* = 3\pi/2$. On the other hand, if $x_0 < 3\pi/2$ the trajectories are attracted to $x_0^* = 3\pi/2$.

Note that the second-order polynomial approximation of the system (63) at the fixed point $x_0^* = \frac{3\pi}{2}$ appears to blow up in finite time for $x_0 > 3\pi/2$. In contrast, the trajectories plotted in Figure 12 exist and remain unique for all times. This discrepancy arises because the Taylor expansion did not include a sufficient number of terms, some of which become increasingly important for stabilizing the behavior of the polynomial approximation as η grows larger.

Example: The dynamical system

$$\frac{dx}{dt} = \underbrace{-x^3}_{f(x)} \tag{69}$$

has a fixed point at $x^* = 0$. Linear stability analysis in this case is ineffective at inferring stability. In fact $f'(x) = -3x^2$, which is equal to zero at $x^* = 0$. The analytical solution to (69) is obtained as

$$\int_{x_0}^{x(t)} \frac{dx}{x^3} = -t \quad \Rightarrow \quad \frac{1}{2}\left(\frac{1}{x(t)^2} - \frac{1}{x_0^2}\right) = t. \tag{70}$$

Therefore,

$$X(t, x_0) = \text{sign}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2 t}}, \tag{71}$$

which shows that $x^* = 0$ is a globally attracting (asymptotically stable) fixed point. This means that $x^* = 0$ attracts all trajectories generated by the ODE (69) independently of the initial condition x_0.

Example: Let ReLU(x) the standard rectified linear unit function. The dynamical system

$$\frac{dx}{dt} = \text{ReLU}(x) \tag{72}$$

has an infinite number of stable fixed points at $x^* < 0$ (none of which is asymptotically stable), and one unstable fixed point at $x^* = 0$.

Lyapunov stability analysis. Lyapunov stability analysis can be used to study the stability of a fixed point x^* without finding the trajectories of (1) (i.e., without solving the differential equation). A typical Lyapunov Theorem has the form: "if there exists a function $V(x)$ that satisfies some conditions on $V(X(t, x_0))$ and $dV(X(t, x_0))/dt$, then the trajectories of the system satisfy some property". A Lyapunov function V can be thought of as generalized potential for a system. The following theorem characterizes asymptotic stability of a fixed point x^*.

Theorem 4. Let $V(x)$ be a continuously differentiable function[14] in a neighborhood of the fixed point x^* satisfying the following properties

a) $V(x)$ has a local minimum at x^*,

b) $V(x)$ does not increase along trajectories of (1), i.e., $dV(X(t, x_0))/dt < 0$, for all x_0 in a neighborhood of x^* (excluding x^*).

Then x^* is is an asymptotically stable fixed point.

Suppose that x_0 is in a neighborhood of x^* then

$$V(x(t_2, x_0)) = V(X(t_1, x_0)) + \int_{t_1}^{t_2} \frac{dV(X(\tau, x_0))}{d\tau} d\tau < V(X(t_1, x_0)) \quad \text{for all } t_2 \geq t_1 \tag{73}$$

Hence, if there exist a function $V(x)$ satisfying a) and b) in Theorem 4 then $X(t, x_0)$ converges monotonically to the local minimum of $V(x)$, which is located at x^* as time increases. How do we construct a function $V(x)$ with the properties stated in Theorem 4? For one-dimensional systems is straightforward to show that property b) is satisfied by the primitive of $-f(x)$ (also known as "potential energy"), i.e.,

$$\frac{dV(x)}{dx} = -f(x). \tag{74}$$

[14] Lyapunov theorems can be proved using only the assumption of continuity, where one says that a Lyapunov function is a continuous function which decreases along every trajectory in a neighborhood of the fixed point x^*.

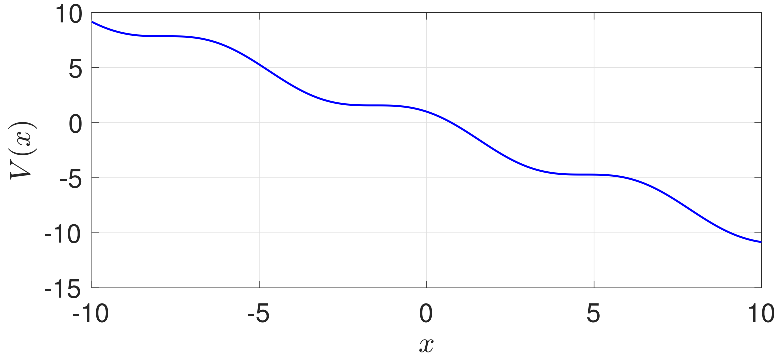


Figure 13: A potential function for the dynamical system (63). Note that this is not a Lyapunov function.

In fact,

$$\frac{dV(X(t,x_0))}{dt} = \frac{dV(X(t,x_0))}{dx}\frac{dX(t,x_0)}{dt} = -f(X(t,x_0))^2 \leq 0. \tag{75}$$

The equality sign holds only at fixed points, which are indeed the only points where $dX(t,x_0)/dt = 0$. If we interpret $f(x)$ as a vector field in the sense described in Figure 10, then $V(x)$ defined in (74) is called *potential energy* for $f(x)$. The potential energy is defined up to an additive constant. Note that the function $V(x)$ defined as in (74) is not necessarily a Lyapunov function as it may not have a local minimum at a fixed point x^*, but rather an inflection point as shown in Figure 13. Of course in one dimension there is not much to learn from the Lyapunov function of system at a fixed point. However, for dynamical systems in n-dimensions we can indeed learn a lot, especially if we are interested in stability analysis of non-hyperbolic fixed points (e.g., in energy-preserving systems).

Remark: The Lyapunov function is not unique to a particular system. Fr example, $V(x) = x^2$ is a Lyapunov function for all one-dimensional systems having a stable node at $x^* = 0$.

Example: Consider the dynamical system (63). A potential for such system is

$$V(x) = V(x_0) - \int_{x_0}^{x} (1 + \sin(y))\, dy = \cos(x) - x + C, \tag{76}$$

where C is a constant. This function is plotted in Figure 13 for $C = 0$. It is seen that $V(x)$ has inflection points at the fixed points of $f(x) = 1 + \sin(x)$, suggesting that such fixed points are half-stable.

Impossibility of trajectory reversals

If there exists a smooth function $V(x)$ satisfying $dV(X(t,x_0))/dt < 0$ then there cannot be any maxima or minima of $X(t,x_0)$ at any finite time t. In particular, this rules out trajectories of the form shown in Figure 14. The proof follows immediately from (73). In fact, for any trajectory reversal there exist t_1 and t_2 such that $X(t_2,x_0) = X(t_1,x_0)$ (see Figure 14). Hence, $V(X(t_2),x_0) = V(X(t_1,x_0))$ which immediately contradicts (73). Note in fact, that $dV(X(t,x_0))/dt$ is not zero and does not change sign in $[t_1,t_2]$).

An alternative method to rule out the possibility of trajectory reversals, such as those depicted in Figure 14, relies on the existence and uniqueness Theorem 1. Since the dynamical system (1) is autonomous, the specific time at which we set the initial condition is irrelevant. This means we are free to translate trajectories left or right in the $(t,x(t))$-plane to generate all possible solutions to the system. However, as illustrated in Figure 15, the existence of a trajectory reversal would violate the conditions of the existence and uniqueness theorem. such behavior would imply that two trajectories intersect, which is not allowed for well-posed initial value problems. Note that this also implies that to compute the flow of one-dimensional

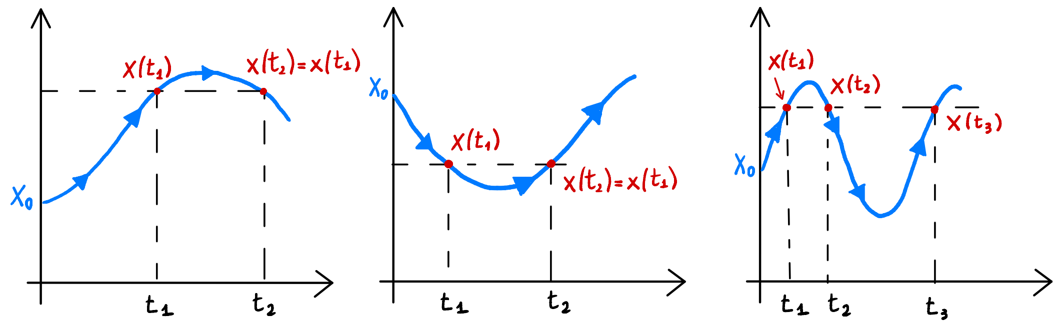


Figure 14: These trajectories are impossible for one-dimensional systems of the form (1).

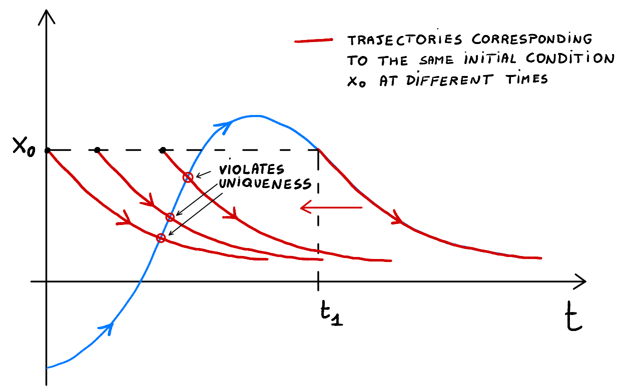


Figure 15: An autonomous dynamical generates trajectories that depend only on x_0 and f. This means that we can translate a trajectory left and right to obtain other solutions of the same system. This translational symmetry, together with the existence and uniqueness Theorem 1, rules out the possibility of trajectory reversals, e.g., the blue trajectory.

systems we just need to compute a few representative trajectories. These can then be translated left or right, as shown in Figure 16 for the system $dx/dt = 1 - x^2$.

Dynamics of one-dimensional systems

In summary, the trajectories of a one-dimensional dynamical system

- Can get to a stable (or half-stable) fixed-point in an infinite time,
- Can blow-up to infinity in a finite or an infinite time,
- Cannot have maxima or minima at any finite time (no overshoot/undershoot, no periodic orbits).

The only attracting sets of one-dimensional dynamical systems are fixed points. In higher dimensions we can have attracting sets that are more complicated, e.g., limit cycles, saddle nodes connected by heteroclinic orbits, strange attractors, etc.

Appendix A: Smoothness of one-dimensional flows

Let us first show that if $f(x)$ in (1) is Lipschitz continuous then the flow $X(t, x_0)$ is continuous with respect to x_0.

Theorem 5. Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$ Lipschitz continuous. Then the solution $X(t, x_0)$ to the initial value

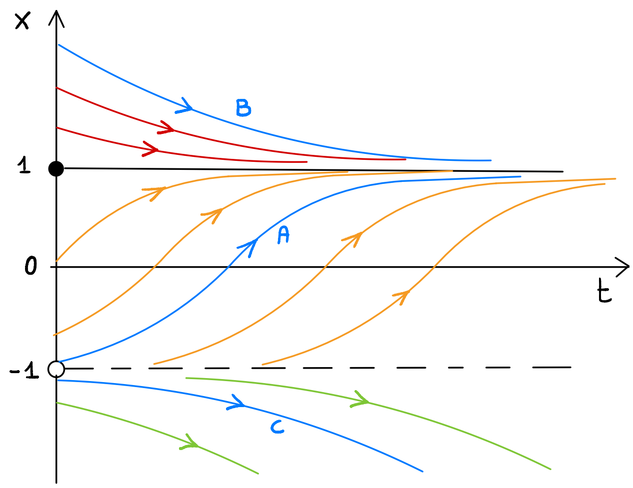


Figure 16: We can use the translational symmetry of the solutions to the autonomous system (1) to construct the entire flow. Specifically, the yellow trajectories are all obtained by translating the trajectory labeled by "A" to the left and to the right. Similarly, the red trajectories are obtained by translating the trajectory labeled as "B" to the left, while the green trajectories are obtained by translating the trajectory labeled by "C" to the left and to the right.

problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0 \tag{77}$$

is continuous with respect to the initial condition x_0.

Proof. Let $X(t, x_0)$ and $X(t, x_0 + \delta)$ be the solutions to the IVP with initial conditions x_0 and $x_0 + \delta$, respectively. Define

$$\eta(t) = X(t, x_0 + \delta) - X(t, x_0).$$

Then

$$\frac{d\eta}{dt} = f(X(t, x_0 + \delta)) - f(X(t, x_0)).$$

Since f is Lipschitz, there exists $L > 0$ such that

$$\left| \frac{d\eta}{dt} \right| \le L|\eta(t)|.$$

By Grönwall's inequality[15] this implies that

$$|\eta(t)| \le |\delta| e^{Lt}.$$

Hence,

$$|X(t, x_0 + \delta) - X(t, x_0)| \le |\delta| e^{Lt} \to 0 \quad \text{as } \delta \to 0.$$

So $X(t, x_0)$ is continuous in x_0. □

[15] Let $\beta(t)$ be a continuous function in $[0, T]$. The Grönwall's lemma says that if for all $t \in [0, T]$

$$\frac{d\eta(t)}{dt} \le \beta(t)\eta(t) \qquad \text{then} \qquad |\eta(t)| \le |\eta(0)| \exp\left[\int_0^T \beta(s) ds\right]. \tag{78}$$

.

Let us now prove Theorem 2, hereafter rewritten for convenience.

Theorem 6. Let $D \subseteq \mathbb{R}$, $f \in C^k(\mathbb{D})$. Then the solution $X(t, x_0)$ to the initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0$$

is of class C^k in x_0.

Proof. Let us assume that $f \in C^1(D)$ and define

$$\theta(t) = \frac{\partial X(t, x_0)}{\partial x_0}. \tag{79}$$

Differentiating the ODE $dX(t, x_0)/dt = f(X(t, x_0))$ with respect to x_0 yields

$$\frac{d\theta}{dt} = f'(X(t, x_0))\theta(t), \quad \theta(0) = 1. \tag{80}$$

This is a linear ODE for θ with smooth coefficient $f'(X(t, x_0))$, since $f \in C^1$ and $X(t, x_0)$ is at least continuous in t and x_0 by Theorem 5. The unique solution is given by

$$\theta(t) = \exp\left(\int_0^t f'(X(s, x_0))\, ds\right). \tag{81}$$

Because the integrand $f'(X(s, x_0))$ is continuous in s and depends continuously on x_0, the function $\theta(t)$ is continuous in both t and x_0. Therefore, the partial derivative $\partial X(t, x_0)/\partial x_0$ exists and is continuous, which implies $X(t, x_0) \in C^1(D)$ with respect to x_0. To obtain higher-order regularity, differentiate the ODE (80), with θ defined in (79) repeatedly using the chain rule. For instance, the second derivative

$$\eta(t) = \frac{\partial^2 X(t, x_0)}{\partial x_0^2}. \tag{82}$$

satisfies

$$\frac{d\eta}{dt} = f''(X(t, x_0))\theta^2(t) + f'(X(t, x_0))\eta(t), \quad \theta_2(0) = 0. \tag{83}$$

This is again a linear (nonhomogeneous) ODE with continuous coefficients, since $f \in C^2$ and $X(t, x_0)$ is at least continuous in x_0 and t. Therefore, by the variation of constant formula a unique solution $\eta(t)$ of the form

$$\eta(t) = \theta(t) \int_0^t f''(X(s, x_0))\theta(s) ds \tag{84}$$

exists and is at least continuous in both t and x_0. This shows that $\partial^2 X(t, x_0)/\partial x_0^2$ exists and is continuous, and so $X(t, x_0) \in C^2(D)$ in x_0. By induction, if $f \in C^k(D)$, each derivative of order $\leq k$ exists and satisfies a linear non-homogeneous ODE with continuous coefficients. Hence, $X(t, x_0) \in C^k(D)$.

□

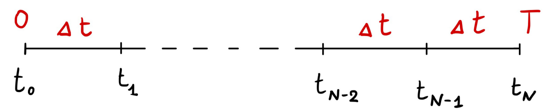
Appendix B: Elementary numerical methods for ODEs

The initial value problem (1) can be equivalently written in an integral form as

$$x(t) = x_0 + \int_0^t \frac{dx(s)}{ds} ds = x_0 + \int_0^t f(x(s)) ds \tag{85}$$

i.e., as an integral equation for $x(s)$. This formulation is quite convenient for developing numerical methods for ODEs based on *numerical quadrature formulas*, i.e., numerical approximations of the temporal integral appearing at the right hand side of (85). For example, consider a discretization of the time interval $[0, T]$ in terms of $N+1$ evenly-spaced time instants

$$t_i = i\Delta t \qquad i = 0, 1, \ldots, N \qquad \text{where} \quad \Delta t = \frac{T}{N}. \tag{86}$$



By applying (85) within each time interval $[t_i, t_{i+1}]$ we obtain

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(x(s))ds. \tag{87}$$

At this point we can approximate the integral at the right hand side if (87), e.g., by using the simple rectangle rule

$$\int_{t_i}^{t_{i+1}} f(x(s))ds \simeq \Delta t f(x(t_i)) \tag{88}$$

This yields the *Euler forward scheme*

$$u_{i+1} = u_i + \Delta t f(u_i), \tag{89}$$

where u_i is an approximation of $x(t_i)$. The Euler forward scheme is an explicit one-step scheme. The adjective "explicit" emphasizes the fact that u_{i+1} can be computed explicitly based on the knowledge of f and u_i using (89). On the other hand, if we approximate the integral at the right hand side of (85) with the trapezoidal rule

$$\int_{t_i}^{t_{i+1}} f(x(s))ds \simeq \frac{\Delta t}{2}\left[f(x(t_{i+1})) + f(x(t_i))\right] \tag{90}$$

we obtain the *Crank-Nicolson scheme*

$$u_{i+1} = u_i + \frac{\Delta t}{2}\left[f(u_i) + f(u_{i+1})\right]. \tag{91}$$

The Crank-Nicolson scheme is "implicit" because the approximate solution at time t_{i+1}, i.e., u_{i+1}, cannot be computed explicitly based on u_i, but requires the solution of a nonlinear equation. Such a solution can be computed numerically by using any method to solve nonlinear equations. These methods are usually iterative, e.g., the bisection method, or the Newton method if f is continuously differentiable. Iterative methods for nonlinear equations can be formulated as fixed point iteration problems. In the specific case of (91) we have

$$u_{i+1} = G(u_{i+1}) \qquad \text{where} \qquad G(u_{i+1}) = u_i + \frac{\Delta t}{2}\left[f(u_i) + f(u_{i+1})\right]. \tag{92}$$

If Δt is small then u_i is close to u_{i+1}. Moreover, if Δt is sufficiently small we have that the Lipschitz constant of G is smaller than 1, which implies that the fixed point iterations will convergence globally to a unique solution u_{i+1} .