Bifurcations of equilibria in one-dimensional dynamical systems

In previous lecture note we studied one-dimensional dynamical systems of the form

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = x_0 \end{cases} \tag{1}$$

and fully characterized their properties. In particular, we proved that trajectory reversals or oscillations are impossible, and that the dynamics is essentially determined by the location of the fixed points and their stability properties. In this course note we study what happens to the locations of the fixed points if $f(x)$ in (1) depends on a parameter $\mu \in \mathbb{R}$, and we are allowed to vary such parameter μ. To this end, we consider the dynamical system[1]

$$\begin{cases} \dfrac{dx}{dt} = f(x,\mu) \\ x(0) = x_0 \end{cases} \tag{2}$$

For each fixed value of μ we can plot $f(x,\mu)$ versus x, and see if there are any fixed points. Equivalently, we can think of $f(x,\mu)$ as a real-valued function in 2 variables, i.e., a surface in the three-dimensional Euclidean space.

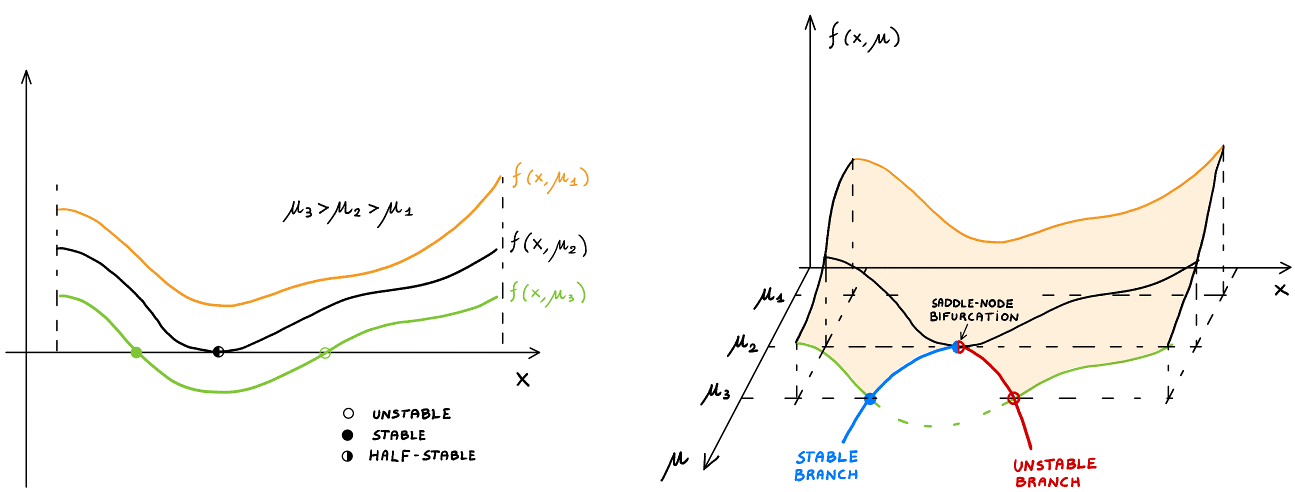

Figure 1: Sketch of a saddle-node bifurcation.

Clearly, the fixed points of the system (2) are in the zero level set[2] of $f(x,\mu)$, i.e., they are solutions of the equation

$$f(x,\mu) = 0. \tag{4}$$

In Figure 1, the zero level set of $f(x,\mu)$ is represented by the stable and unstable branches of fixed points that originate from a saddle-node bifurcation at $\mu = \mu_2$. Of course, there are many other ways the function $f(x,\mu)$ can intersect the plane (x,μ). For instance, we can have the zero level set corresponding to the so-called sub-critical pitchfork bifurcation. This is sketched in Figure 2.

[1] More generally, $f(x)$ can depend on multiple parameters, i.e., we can have $f(x,\mu_1,\ldots,\mu_M)$ in equation (2).

[2] The zero level set of a function $f(x,\mu)$ is the set of points $(x,\mu) \in \mathbb{R}^2$ such that the function is equal to zero, i.e.,

$$\{(x,\mu) \in \mathbb{R}^2 : \; f(x,\mu) = 0\} \qquad \text{(zero level set of } f). \tag{3}$$

If the function $f(x,\mu)$ does not intersect the (x,μ) plane, then the zero level set is empty.

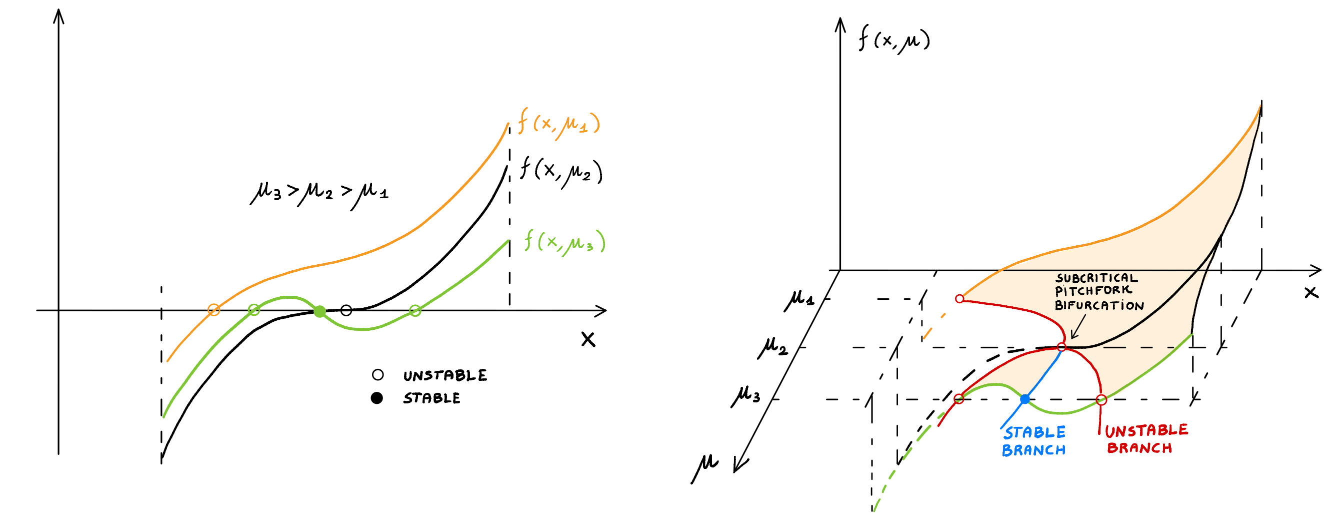

Figure 2: Sketch of a subcritical pitchfork bifurcation.

Bifurcation diagram. The zero level set of $f(x, \mu)$, i.e., the set of points $(x, \mu) \in \mathbb{R}^2$ satisfying (4) is called *bifurcation diagram* of fixed points. In practice, the bifurcation diagram provides the location of all fixed points of the system as a function of the parameter μ. Hereafter we sketch the bifurcation diagrams corresponding to the saddle-node bifurcation sketched in Figure 1, and the subcritical pitchfork bifurcation sketched Figure 2.

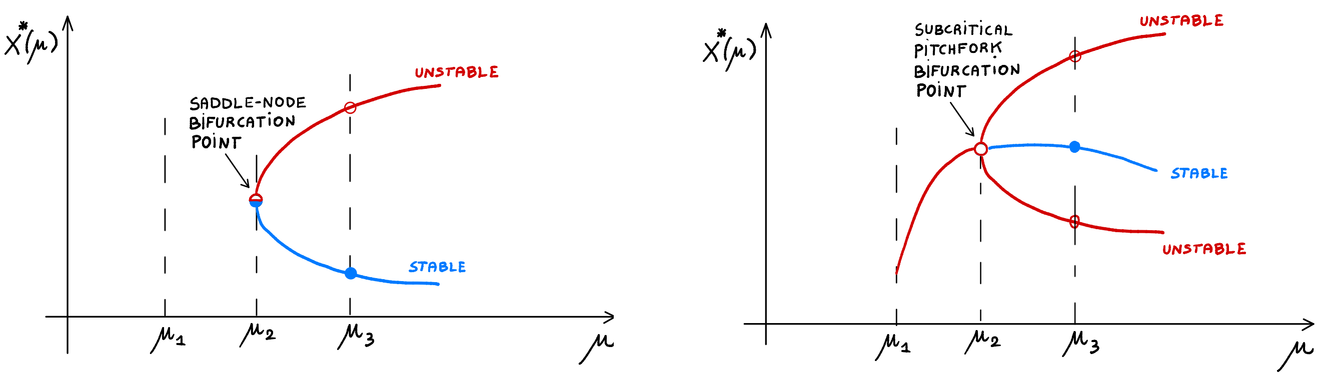

Figure 3: Bifurcation diagrams corresponding to the saddle-node bifurcation sketched in Figure 1, and the subcritical pitchfork bifurcation sketched in Figure 2.

In the bifurcation diagram we usually plot the location of the fixed points $x^*(\mu)$ as a function of the bifurcation parameter μ, but it is also possible to plot the bifurcation parameter $\mu(x^*)$ versus the location of the fixed points x^*. In this setting, the saddle-node bifurcation diagram shown in Figure 3(left) becomes a parabolic function $\mu(x^*)$ with upward concavity.

What is the relation between the coordinates of the fixed points x^* and the parameter μ? In particular, is it possible to express the zero level set of $f(x, \mu)$ as a graph of a smooth function? The answer is provided by the implicit function theorem.

Theorem 1 (Implicit function theorem)**.** Let $f(x,\mu)$ be a function of class $\mathcal{C}^1$ (i.e., continuously differentiable) in x and μ in a neighborhood of a point (x^*,μ^*) such that $f(x^*,\mu^*)=0$. If

$$\frac{\partial f(x^*,\mu^*)}{\partial x}\neq 0 \tag{5}$$

then there exists a neighborhood B of μ^* in which the zero level set of $f(x,\mu)$ can be represented as a graph of a smooth function $x^*(\mu)$, i.e.,

$$f(x^*(\mu),\mu)=0 \qquad \text{for all } \mu\in B. \tag{6}$$

The function $x^*(\mu)$ is of class $\mathcal{C}^1(B)$ (continuously differentiable in B) and it satisfies the additional properties:

$$x^*(\mu^*)=x^*, \qquad\qquad \frac{dx^*(\mu^*)}{d\mu}=-\frac{\dfrac{\partial f(x^*,\mu^*)}{\partial \mu}}{\dfrac{\partial f(x^*,\mu^*)}{\partial x}}. \tag{7}$$

Comments on the implicit function theorem:

- Theorem 1 indicates that the bifurcation diagram is composed of smooth curves $x^*(\mu)$ (continuously differentiable in μ), except at points (x^*,μ^*) where

$$\frac{\partial f(x^*,\mu^*)}{\partial x}=0. \tag{8}$$

- Property (7) follows immediately by differentiating (6) with respect to μ and evaluating the derivative at at $\mu=\mu^*$. In fact,

$$f(x^*(\mu),\mu)=0 \quad\Rightarrow\quad \frac{df(x^*(\mu),\mu)}{d\mu}=0 \quad\Rightarrow\quad \frac{\partial f(x^*(\mu),\mu)}{\partial\mu}+\frac{\partial f(x^*(\mu),\mu)}{\partial x}\frac{dx^*(\mu)}{d\mu}=0. \tag{9}$$

  By evaluating the last equation at $\mu=\mu^*$ and recalling that $x^*(\mu^*)=x^*$ we obtain (7). Note that we can divide by $\partial f(x^*,\mu^*)/\partial x$ because it is nonzero by assumption (5).

- The role of x and μ can be reversed in Theorem 1. In other words, it is possible to formulate the implicit function theorem by choosing x as independent variable and μ as dependent variable. In this formulation, if (x^*,μ^*) is in the zero level set of f and $\partial f(x^*,\mu^*)/\partial\mu\neq 0$ then there exists a smooth function $\mu^*(x)$ in a neighborhood of x^* that represents the zero level set of $f(x,\mu)$ for all x in such a neighborhood, i.e., $f(x,\mu^*(x))=0$. With reference to the saddle-node bifurcation sketched in Figure 3, it is seen that at the saddle-node bifurcation point we have $dx^*(\mu^*)/d\mu=\infty$, suggesting that $\partial f(x^*,\mu^*)/\partial x=0$ and $\partial f(x^*,\mu^*)/\partial\mu\neq 0$ (see Eq. (7)). On the other hand, $d\mu^*(x^*)/dx=0$, implying again that $\partial f(x^*,\mu^*)/\partial x=0$.

From the discussion above, it appears that the conditions

$$f(x^*,\mu^*)=0, \qquad \frac{\partial f(x^*,\mu^*)}{\partial x}=0 \tag{10}$$

may suggest that a bifurcation is taking place at (x^*,μ^*). While these condition are indeed necessary (otherwise the implicit function theorem applies), they are not sufficient to guarantee the existence of a bifurcation as the following example clearly demonstrates.

Example: Consider the function

$$f(x,\mu)=x^3+\mu. \tag{11}$$

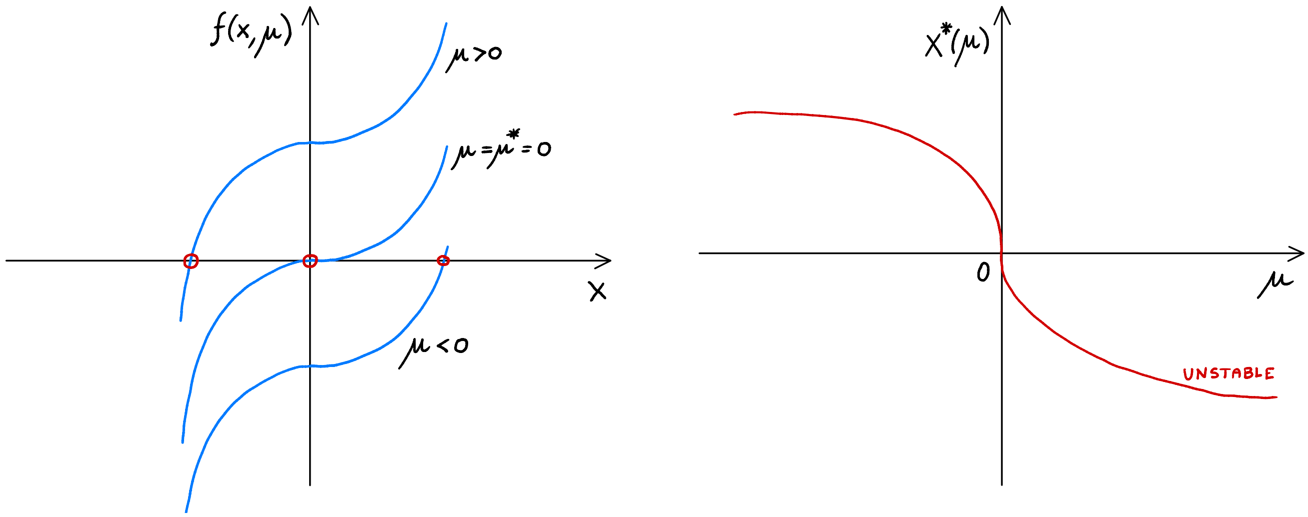

Figure 4: Sketch of the function (11) for different μ (left) and bifurcation diagram (right). In this case there is only one unstable fixed point that moves around as we very the parameter μ, and no bifurcation whatsoever. Yet, the (necessary) conditions for a bifurcation summarized in equation (10) are both satisfied at $(x^*, \mu^*) = (0, 0)$.

Clearly

$$f(0,0) = 0, \qquad \frac{\partial f(0,0)}{\partial x} = 2x^2\big|_{(x,\mu)=(0,0)} = 0. \tag{12}$$

Hence, $(x^*, \mu^*) = (0, 0)$ may be a bifurcation point. However, the zero level set in this case can be expressed analytically as (see Figure 4)

$$f(x,\mu) = x^3 + \mu = 0 \qquad \Rightarrow \qquad x^*(\mu) = -\sqrt[3]{\mu}. \tag{13}$$

This shows that there is indeed no bifurcation at $(x^*, \mu^*) = (0, 0)$. In Figure 4 we sketch the function (11) for different values of μ, and the corresponding bifurcation diagram.

In general, a bifurcation is characterized by two or more branches of fixed point intersecting at some location for some value of the bifurcation parameter μ (see Figure 5). Such multiplicity of branches emanating from the bifurcation point (x^*, μ^*) is usually associated with non-invertibility of the zero level set of $f(x, \mu)$ at (x^*, μ^*), which can be studied by using Taylor series (next section). Of course, it is possible to have functions $f(x, \mu)$ with rather complicated zero level sets, and multiple bifurcations of different types. For example, in Figure 5 we sketch a bifurcation diagram with five different types of bifurcations. In particular,

- Saddle-node bifurcation,
- Transcritical bifurcation,
- Pitchfork bifurcation (supercritical and subcritical).

The "exotic" bifurcation looks like a saddle-node but it involves four branches instead of two.

It is important to always keep in mind that the bifurcation diagram represents the location of the fixed points of the systems as a function of μ. The continuity requirement we imposed on $f(x, \mu)$ prohibits fixed points of the same type, e.g., two stable nodes, to face each other. Consequently, two stable branches or two unstable branches cannot be facing each other in the bifurcation diagram.

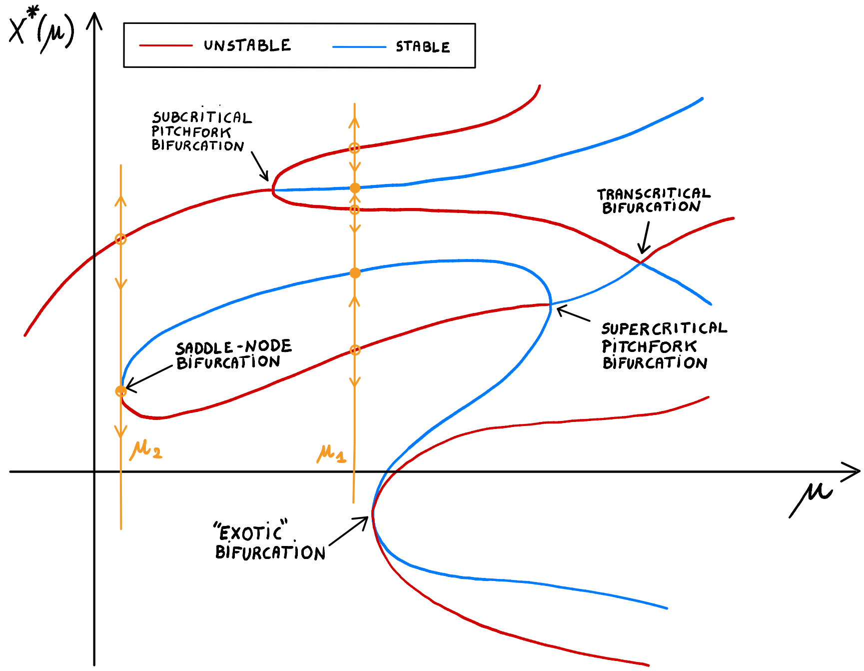

Figure 5: Sketch of a bifurcation diagram with 5 different bifurcations.

Polynomial approximation at bifurcation points. To study invertibility of the zero level set of $f(x,\mu)$ in a neighborhood of a point (x^*,μ^*) that belongs to such zero level set we expand $f(x,\mu)$ in a Taylor series

$$f(x,\mu) = \sum_{k,m=0}^{\infty} \frac{1}{k!m!}\frac{\partial^{k+m} f(x^*,\mu^*)}{\partial x^k \partial \mu^m}(x-x^*)^k(\mu-\mu^*)^m, \tag{14}$$

i.e.,

$$\begin{aligned} f(x,\mu) =& f(x^*,\mu^*) + \frac{\partial f(x^*,\mu^*)}{\partial x}(x-x^*) + \frac{\partial f(x^*,\mu^*)}{\partial \mu}(\mu-\mu^*) + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial x^2}(x-x^*)^2 + \\ & \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial \mu^2}(\mu-\mu^*)^2 + \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu}(x-x^*)(\mu-\mu^*) + \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3}(x-x^*)^3 + \\ & \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial \mu^3}(\mu-\mu^*)^3 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x \partial \mu^2}(x-x^*)(\mu-\mu^*)^2 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^2 \partial \mu}(x-x^*)^2(\mu-\mu^*) + \cdots \end{aligned} \tag{15}$$

We know that if (x^*,μ^*) is a bifurcation point then

$$f(x^*,\mu^*) = 0, \qquad \text{and} \qquad \frac{\partial f(x^*,\mu^*)}{\partial x} = 0. \tag{16}$$

A substitution of (16) into (15) yields

$$\begin{aligned} f(x,\mu) =& \frac{\partial f(x^*,\mu^*)}{\partial \mu}R + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial x^2}X^2 + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial \mu^2}R^2 + \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu}XR + \\ & \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3}X^3 + \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial \mu^3}R^3 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x \partial \mu^2}XR^2 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^2 \partial \mu}X^2R + \cdots \end{aligned} \tag{17}$$

where we defined the "centered" variables

$$X = x - x^*, \qquad R = \mu - \mu^*. \tag{18}$$

If (x, μ) is also in the zero level set then $f(x, \mu) = 0$ in (17). This yields polynomial equation in X and R that characterizes the locations of the fixed points in a neighborhood of a fixed point (x^*, μ^*) satisfying (16). The multiplicity of possible solutions to such polynomial equation is what eventually yields multiple branches of equilibria emanating from the fixed point (x^*, μ^*).

Depending on the leading order of the polynomial expansion of the system at the bifurcation point, we can have a different number of branches of equilibria involved in the bifurcation process. As we shall see hereafter, both saddle-node and transcritical bifurcations are represented locally by polynomials of degree 2. Pitchfork bifurcations by polynomials of degree 3. The "exotic" bifurcation shown in Figure 5 is generated by system that is locally equivalent to a polynomial of degree 4.

Saddle-node bifurcation. We have now all element to characterize the saddle-node bifurcation sketched in Figure 1 and Figure 3(left).

Theorem 2 (Saddle-node bifurcation)**.** Let (x^*, μ^*) be a fixed point of the dynamical system (2), i.e., $f(x^*, \mu^*) = 0$. If the following conditions are satisfied

$$\frac{\partial f(x^*, \mu^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*, \mu^*)}{\partial \mu} \neq 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} \neq 0, \tag{19}$$

then the system undergoes a saddle-node bifurcation at (x^*, μ^*).

To characterize the saddle-node bifurcation *quantitatively*, we choose R in (17) to be of the same order of magnitude as X^2. For example, if $X \simeq 10^{-5}$ then $R \simeq 10^{-10}$. In this way, the leading terms in the Taylor series (17) have the same order of magnitude, and we can neglect higher-order terms in a straightforward way. In this assumption, the Taylor series (17) can be written as

$$f(x, \mu) = AX^2 + BR + \cdots . \tag{20}$$

where we set

$$A = \frac{\partial f(x^*, \mu^*)}{\partial \mu} \neq 0, \qquad \text{and} \qquad B = \frac{1}{2}\frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} \neq 0. \tag{21}$$

Equation (20) allows us to write the following polynomial approximation[3] of the dynamical system (2) in a neighborhood of the bifurcation point (x^*, μ^*)

$$\frac{dX}{dt} = AX^2 + BR. \tag{23}$$

Diving by A and rescaling the time variable t as $\tau = At$ yields[4]

$$\frac{dX}{d\tau} = X^2 + H \qquad \text{(normal form)} \tag{25}$$

where $\tau = At$, and $H = RB/A$ is a rescaled version of the bifurcation parameter μ. Any dynamical system that undergoes a saddle-node bifurcation can be written as (25) in a neighborhood of the bifurcation point, i.e., for (x, μ) very close to (x^*, μ^*). This is the reason why (25) is called the *normal form* of a dynamical system that undergoes a saddle-node bifurcation.

In Figure 6 we clarify the meaning of the normal form of a saddle-node bifurcation. In Figure 7 we sketch the corresponding bifurcation diagram.

[3]Note that since x^* is a constant we have

$$\frac{dX}{dt} = \frac{d(x - x^*)}{dt} = \frac{dx}{dt}. \tag{22}$$

[4]In equation (25) we assumed that $A > 0$. If $A <$ then we divide by the modulus of A, i.e., $|A|$, which leaves a minus sign in front of X^2 in (25), i.e.,

$$\frac{dX}{d\tau} = -X^2 + H \qquad , \qquad \tau = |A|t, \qquad , H = RB/|A|. \tag{24}$$

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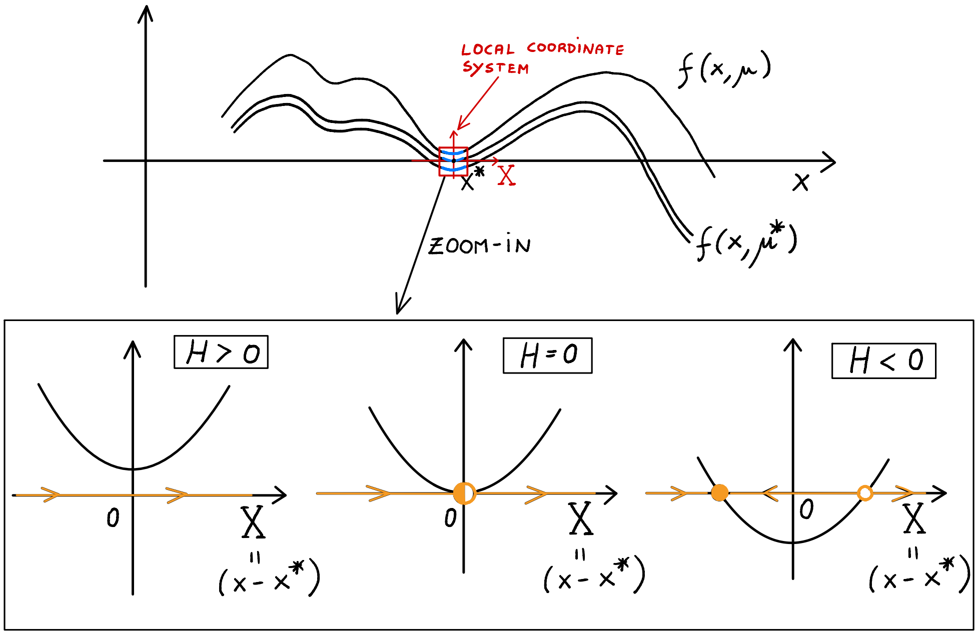


Figure 6: Saddle-node bifurcation. Shown is the function $f(x, \mu)$ for three values of μ, and a zoom-in of the bifurcation process. The nonlinear dynamical system (2) in such a small region is approximated by the normal form (25) (after appropriate rescaling).

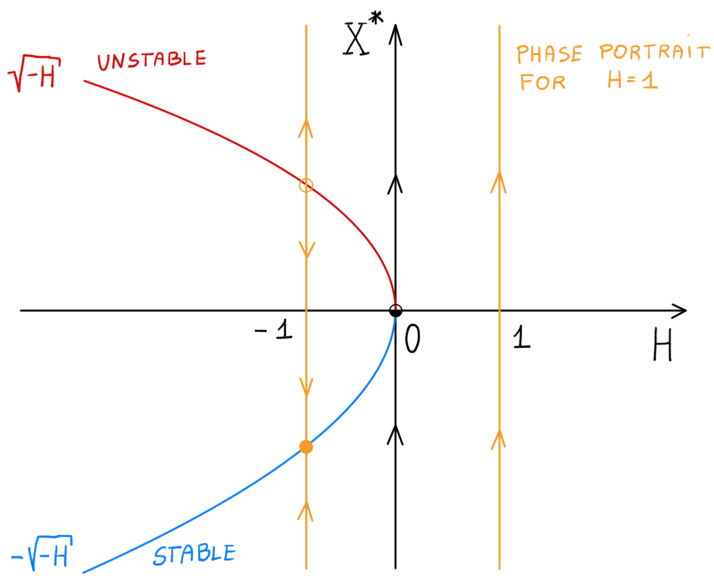


Figure 7: Bifurcation diagram for the normal form of a saddle-node bifurcation. Similarly to Figure 6, this bifurcation diagram describes what happens in an extremely small region that includes the bifurcation point (x^*, μ^*), i.e., the region in red in Figure 6.

Example: Consider the nonlinear system

$$\frac{dx}{dt} = \sin(x) + \mu \tag{26}$$

In Figure 8 we plot $f(x, \mu)$ together with its zero level set, i.e., the bifurcation diagram. Note that if $\mu^* = 1$

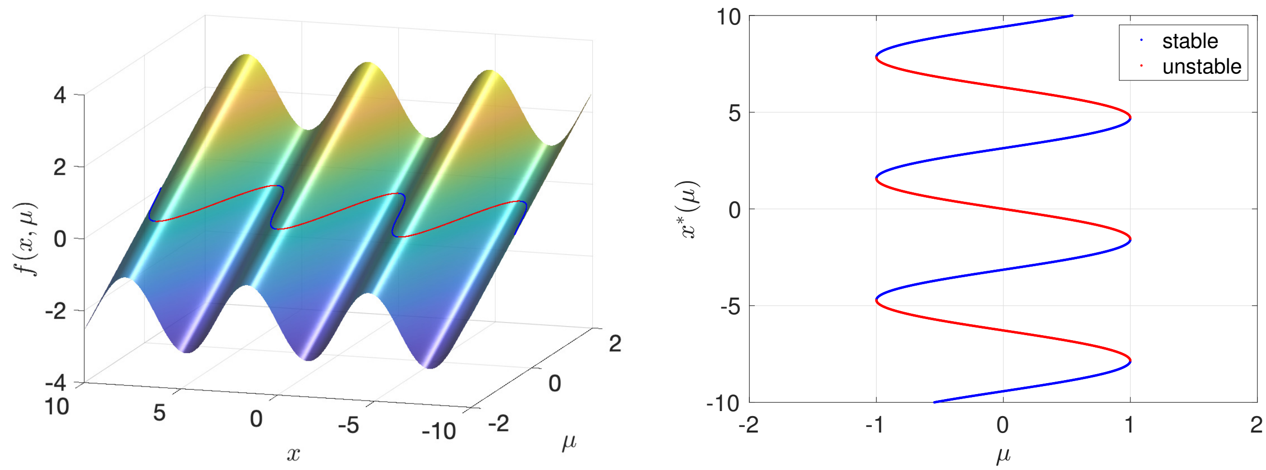


Figure 8: Plot of the right hand side of equation 26, i.e., $f(x,\mu) = \sin(x) + \mu$ together with its zero level set (left), and bifurcation diagram (right).

we have that $f(x,\mu) = \sin(x) + \mu$ is tangent to the x axis at the points

$$x_k^* = -\frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}. \tag{27}$$

At such points we have

$$\frac{\partial f(x_k^*,\mu^*)}{\partial x} = \cos(x_k^*) = 0 \qquad \frac{\partial f(x_k^*,\mu^*)}{\partial \mu} = 1 \neq 0 \qquad \frac{\partial^2 f(x_k^*,\mu^*)}{\partial x^2} = -\sin(x_k^*) = -1 \neq 0. \tag{28}$$

Hence, the conditions of Theorem 2 are satisfied, implying that (x_k^*,μ^*) $(k \in \mathbb{Z})$ are all saddle-node bifurcations. It is straightforward to show that when $\mu^* = -1$ there is another infinite number of of saddle-node bifurcations at

$$x_k^* = \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}. \tag{29}$$

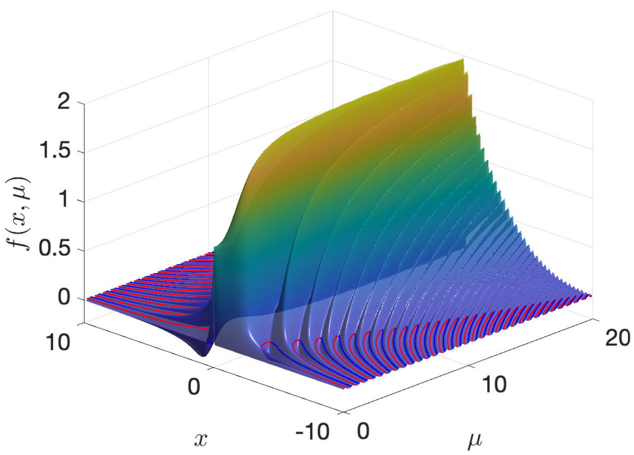
Example: Consider the nonlinear system

$$\frac{dx}{dt} = e^{-x^2/\mu} - \frac{\sin(x\mu)}{(x^2+1)}. \tag{30}$$

In this case, it is not possible to determine the fixed points of the system analytically. In fact, the fixed points are solutions to the transcendental equation

$$e^{-x^2/\mu} = \frac{\sin(x\mu)}{(x^2+1)}, \tag{31}$$

which cannot be solved analytically. However, it is rather straightforward to compute the fixed points numerically, e.g., as zero level sets of the two dimensional function (30) or using any root finding solver. The result is shown in Figure 9, where we see that there is an infinite number of saddle node bifurcations that tend to cluster as μ increases



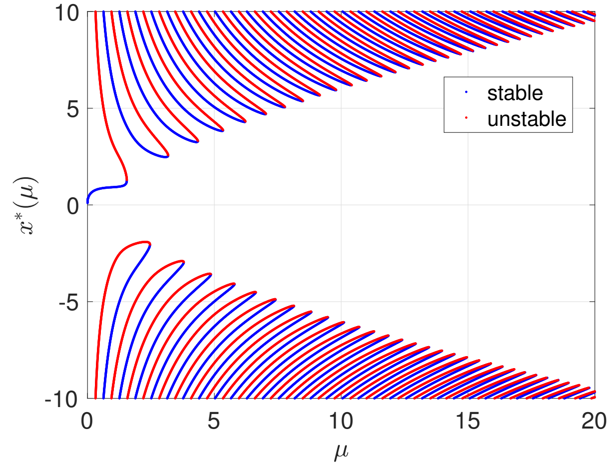


Figure 9: Plot of $f(x, \mu)$ defined in equation 30 (right hand side) together with its zero level set (left), and bifurcation diagram (right).

Transcritical bifurcation. Transcritical bifurcations are rather common bifurcations of equilibria, which can be characterized by the following theorem.

Theorem 3 (Transcritical bifurcation)**.** Let (x^*, μ^*) be a fixed point of the dynamical system (2), i.e., $f(x^*, \mu^*) = 0$. If the following conditions are satisfied

$$\frac{\partial f(x^*, \mu^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*, \mu^*)}{\partial \mu} = 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} \neq 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} \neq 0 \tag{32}$$

then the system undergoes a transcritical bifurcation at (x^*, μ^*).

A substitution of (32) into (17) yields (to leading order in $X = x - x^*$ and $R = \mu - \mu^*$)

$$f(x, \mu) = BX^2 + CXR + \cdots . \tag{33}$$

In equation (33) we set

$$B = \frac{1}{2}\frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} \neq 0, \qquad \text{and} \qquad C = \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} \neq 0. \tag{34}$$

Hence, the dynamics nearby a transcritical bifurcation point is characterized by the following polynomial approximation of the dynamical system (2)

$$\frac{dX}{dt} = BX^2 + CXR, . \tag{35}$$

which can be normalized (divide by B) as[5]

$$\frac{dX}{d\tau} = X^2 + XH \qquad \text{(normal form)}, \tag{37}$$

where $\tau = Bt$, and $H = CR/B$. The fixed points of (37) are $X^* = 0$ and $X^* = -H$. In Figure 10 plot the velocity vector that defines the normal form of a transcritical bifurcation and sketch the bifurcation diagram.

[5] As in the case of the saddle-node bifurcation, if $B < 0$ then we divide by the modulus of B, which yields a minus in front of X^2 in (35), i.e.,

$$\frac{dX}{d\tau} = -X^2 + XH. \tag{36}$$

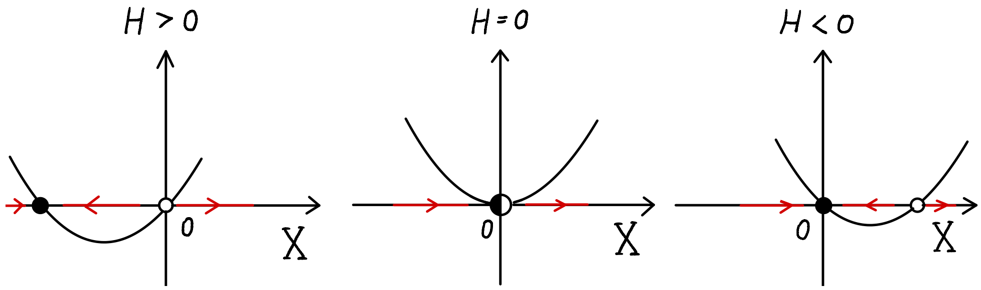


Figure 10: Transcritical bifurcation in local coordinates. Shown is the function $X^2 + XH$ appearing at the right hand side of the normal form (37) for three values of H. The nonlinear dynamical system (2) is approximated by the normal form (37) in a neighborhood of the bifurcation point (after appropriate rescaling).

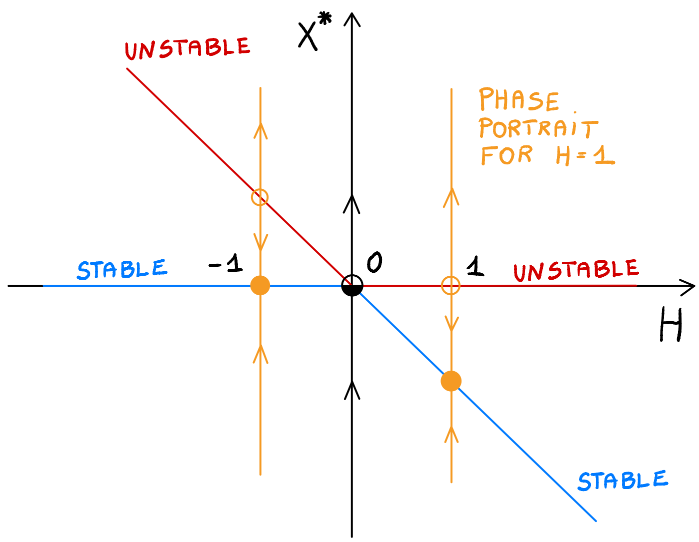


Figure 11: Bifurcation diagram for the normal form of a transcritical bifurcation.

Example: Consider the following dynamical system

$$\frac{dx}{dt} = \underbrace{\mu \log(x) + x - 1}_{f(x,\mu)}. \tag{38}$$

The fixed points are obtained by setting $f(x, \mu) = 0$. This yields,

$$\mu \log(x) = 1 - x. \tag{39}$$

Clearly, for $x = 1$ the equation above reads $0 = 0$, which means that $x^* = 1$ is a fixed point for all values of μ. Next, we compute the derivative of $f(x, \mu)$ with respect to x

$$\frac{\partial f(x, \mu)}{\partial x} = \frac{\mu}{x} + 1. \tag{40}$$

A necessary condition for (x^*, μ^*) to be a bifurcation point is

$$\frac{\partial f(x^*, \mu^*)}{\partial x} = 0 \qquad \Rightarrow \qquad \mu^* = -x^*. \tag{41}$$

Recalling that $x^* = 1$ is always a fixed point, we find that $(x^*, \mu^*) = (1, -1)$ could be a bifurcation point. Let us verify that $(1, -1)$ is indeed a transcritical bifurcation point. To this end, we just need to verify the conditions in Theorem 3. We have,

$$\frac{\partial f(1,-1)}{\partial \mu} = 0, \qquad \frac{\partial^2 f(1,-1)}{\partial x^2} = 1 \neq 0, \qquad \frac{\partial^2 f(1,-1)}{\partial x \partial \mu} = 1 \neq 0. \tag{42}$$

Therefore $(x^*, \mu^*) = (1, -1)$ is a transcritical bifurcation point. Let us compute the normal form of the system (38) at the bifurcation point. Recalling (34)-(35) and using (42) we have

$$\frac{dX}{dt} = \frac{X^2}{2} + XR, \tag{43}$$

where $X = x - 1$ and $R = \mu + 1$. Divide (43) by 1/2 to obtain the normal form

$$\frac{dX}{d\tau} = X^2 + XH, \tag{44}$$

where

$$\tau = \frac{t}{2}, \qquad \text{and} \qquad H = 2(\mu + 1). \tag{45}$$

Pitchfork bifurcation. Another type of rather common bifurcation of equilibria is the pitchfork bifurcation, which can be *supercritical* or *subcrititical* (see Figure 2 and Figure 5). The following Theorem characterizes pitchfork bifurcations.

Theorem 4 (Pitchfork bifurcation)**.** Let (x^*, μ^*) be a fixed point of the dynamical system (2), i.e., $f(x^*, \mu^*) = 0$. If the following conditions are satisfied

$$\frac{\partial f(x^*,\mu^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*,\mu^*)}{\partial \mu} = 0, \qquad \frac{\partial^2 f(x^*,\mu^*)}{\partial x^2} = 0, \qquad \frac{\partial^3 f(x^*,\mu^*)}{\partial x^3} \neq 0, \qquad \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu} \neq 0 \tag{46}$$

then the system undergoes a pitchfork bifurcation at (x^*, μ^*).

As mentioned above, pitchfork bifurcations can be supercritical or subcritical, depending on the sign of $\partial^3 f/\partial x^3$ and $\partial^2 f/\partial x \partial \mu$. A substitution of (46) into (17) yields (to leading order in $X = x - x^*$ and $R = \mu - \mu^*$

$$f(x,\mu) = DX^3 + CXR + \cdots . \tag{47}$$

where we set

$$D = \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3} \neq 0, \qquad \text{and} \qquad C = \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu} \neq 0. \tag{48}$$

Hence, to leading order, we obtain the following polynomial approximation of (2) at a pitchfork bifurcation point

$$\frac{dX}{dt} = DX^3 + CXR. \tag{49}$$

Dividing by the modulus of D yields

$$\frac{dX}{d\tau} = X^3 + XH \qquad (D > 0) \qquad \text{subcritical pitchfork}, \tag{50}$$

$$\frac{dX}{d\tau} = -X^3 + XH \qquad (D < 0) \qquad \text{supercritical pitchfork}, \tag{51}$$

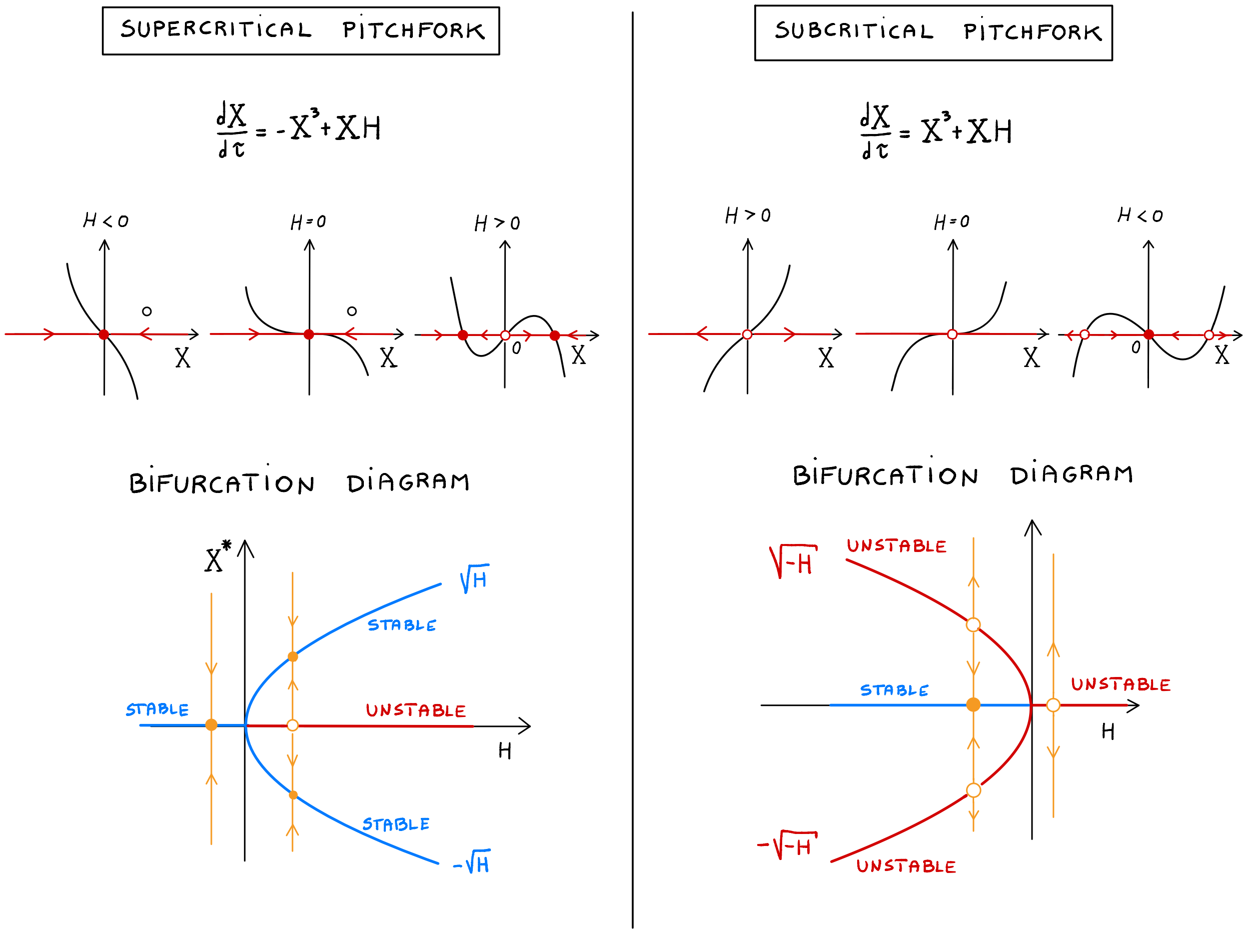


Figure 12: Bifurcation diagrams for supercritical and subcritical pitchfork bifurcations.

where we set $\tau = |D|t$ and $H = CR/|D|$. Any dynamical system of the form (2) that undergoes a pitchfork bifurcation at (x^*, μ^*) can be written (to leading order in X and R after appropriate rescaling) either as (50) or (51) in a neighborhood of (x^*, μ^*). For this reason, (50) and (51) are referred to as the em normal forms of the supercritical and subcritical pitchfork bifurcations, respectively. In Figure 12 plot the velocity vectors associated with the normal forms (50) or (51) and sketch the bifurcation diagrams.

Example: Consider the system

$$\frac{dx}{dt} = \underbrace{\sin(x) + \mu x}_{f(x,\mu)}. \tag{52}$$

The fixed points are solutions to the transcendental equation

$$\sin(x) + \mu x = 0. \tag{53}$$

Clearly $x^* = 0$ is a fixed point for all μ. Note also that for $x \neq 0$ the bifurcation diagram is completely defined by the equation

$$\mu(x^*) = \frac{\sin(x^*)}{x^*}, \tag{54}$$

which explicitly expresses the bifurcation parameter as a function of the location of the fixed points. The derivative of the right hand side of (52) is

$$\frac{\partial f}{\partial x} = \cos(x) + \mu. \tag{55}$$

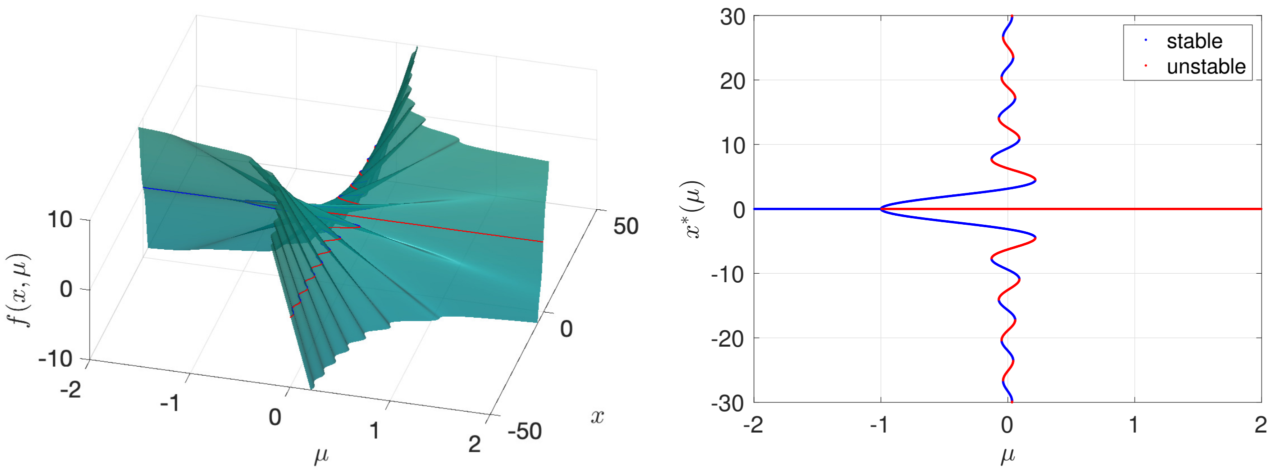


Figure 13: Bifurcation diagram for the system (52).

Hence, for $x^* = 0$ we have that $\mu^* = -1$ makes (55) equal to zero. This means that $(x^*, \mu^*) = (0, -1)$ could be a bifurcation point as it satisfies the necessary conditions (10). Let us verify that $(x^*, \mu^*) = (0, -1)$ is indeed a pitchfork bifurcation point. To this end, it is necessary and sufficient to verify the conditions in Theorem 4. We have

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x) \quad \Rightarrow \frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} = 0, \tag{56}$$

$$\frac{\partial^3 f}{\partial x^3} = -\cos(x) \quad \Rightarrow \frac{\partial^3 f(x^*, \mu^*)}{\partial x^3} = -1, \tag{57}$$

$$\frac{\partial^2 f}{\partial x \partial \mu} = 1 \quad \Rightarrow \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} = 1. \tag{58}$$

Hence, there is a *supercritical pitchfork bifurcation* point at $(x^*, \mu^*) = (0, -1)$. Note, in fact, that (57) implies that the coefficient D in (48) is negative, and therefore the normal form representing the bifurcation in this case is (51). As shown in Figure 13, the system exhibits also an infinite number of saddle-node bifurcations as the parameter μ is varied. Such saddle-node bifurcations are defined analytically by equation (54).

Other bifurcations of equilibria. The Taylor series (17) can be (to leading order) a rather arbitrary polynomial in X and R. This opens the possibility to have more "exotic" bifurcations of equilibria in which one point splits into four points or more (see Figure 5), or bifurcation in which multiple stable and unstable branches intersect at one point.