Introduction to n-dimensional dynamical systems

Consider the following n-dimensional system of nonlinear ODEs

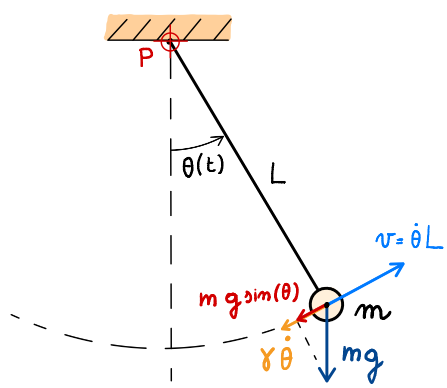
$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t) \cdots x_n(t)]^T$ is a vector of phase variables, $\boldsymbol{f} : D \to \mathbb{R}^n$, and D is a subset of $\mathbb{R}^n$. In an expanded notation the system (1) can be written as

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, \ldots, x_n) \\ \dfrac{dx_2}{dt} = f_2(x_1, \ldots, x_n) \\ \quad\vdots \\ \dfrac{dx_n}{dt} = f_n(x_1, \ldots, x_n) \\ x_1(0) = x_{10} \\ x_2(0) = x_{20} \\ \quad\vdots \\ x_n(0) = x_{n0} \end{cases} \tag{2}$$

Initial value problems of the form (1) can model many physical systems. Let us provide a few examples.

Example: Consider the following idealized pendulum (point mass), subject to gravity and viscous friction



As is well-known from physics (Newton's law), the rate of change (time derivative) of the angular momentum of the point mass m relative to the point P equals the momentum of the external forces acting on the point mass. The external forces in this case are gravity and viscous friction. Setting up the balance of momenta yields

$$mL^2 \frac{d^2\theta}{dt^2} = -mgL\sin(\theta) - \gamma L \frac{d\theta}{dt}, \tag{3}$$

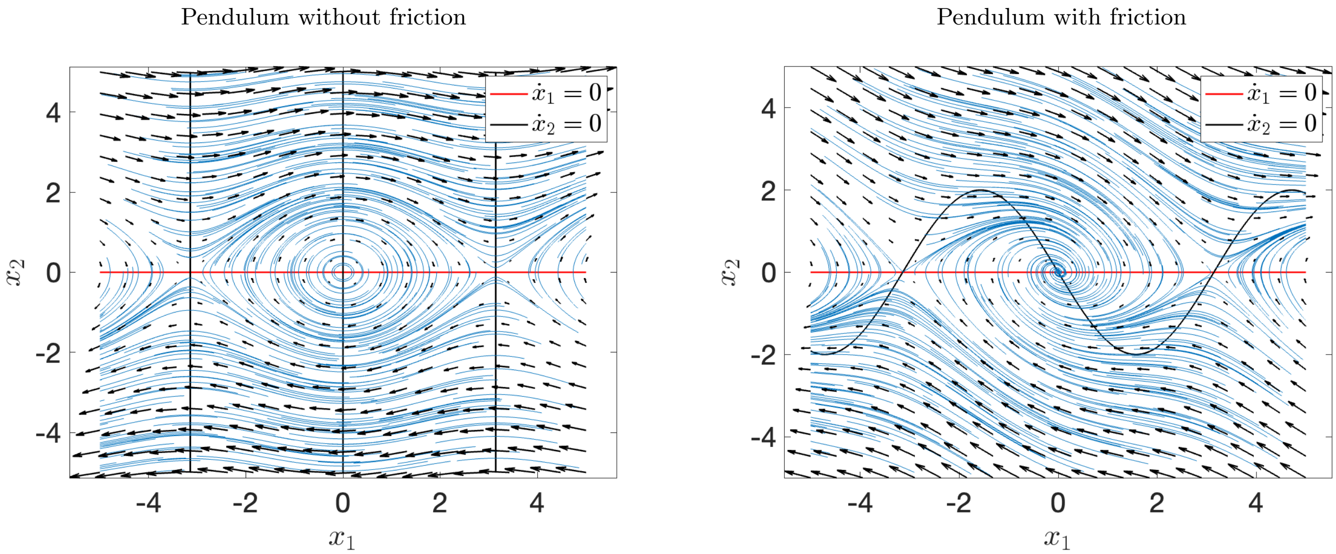


Figure 1: Flow (phase portrait) generated by the the pendulum equations (8) for $\{g/L, \gamma/(Lm)\} = \{(1, 0)\}$ (a) and for $\{g/L, \gamma/(Lm)\} = \{(1, 0.05)\}$ (b). In the latter case the oscillations of the pendulum decay asymptotically to zero due to the viscosity γ.

i.e.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta) - \frac{\gamma}{Lm}\frac{d\theta}{dt}. \tag{4}$$

This is a second-order nonlinear ordinary differential equation in $\theta(t)$. The equation can be easily transformed into a system two first-order nonlinear ODEs by defining the new variables

$$x_1(t) = \theta(t), \qquad x_2(t) = \frac{d\theta(t)}{dt}. \tag{5}$$

Based on the definition of $x_1(t)$ and $x_2(t)$ we have

$$\frac{dx_1}{dt} = x_2. \tag{6}$$

Moreover, by differentiating $x_2(t)$ with respect to time and using equation (4) we obtain

$$\frac{dx_2}{dt} = -\frac{g}{L}\sin(x_1) - \frac{\gamma}{Lm}x_2. \tag{7}$$

Hence, equation (4) can be represented as a system of two first-order ODEs

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = -\dfrac{g}{L}\sin(x_1) - \dfrac{\gamma}{Lm}x_2, \end{cases} \tag{8}$$

i.e., the dynamics of the pendulum can be described by a two-dimensional dynamical system. To solve the system of ODEs (8), we need an initial condition for the position of the pendulum $x_1(0)$, and an initial condition for the velocity of the pendulum $x_2(0)$. In Figure 1 we plot the trajectories of the pendulum corresponding to different initial conditions. Given the nature of the phase space coordinates $x_1(t)$ and $x_2(t)$, it is clear that the dynamics of the the pendulum un

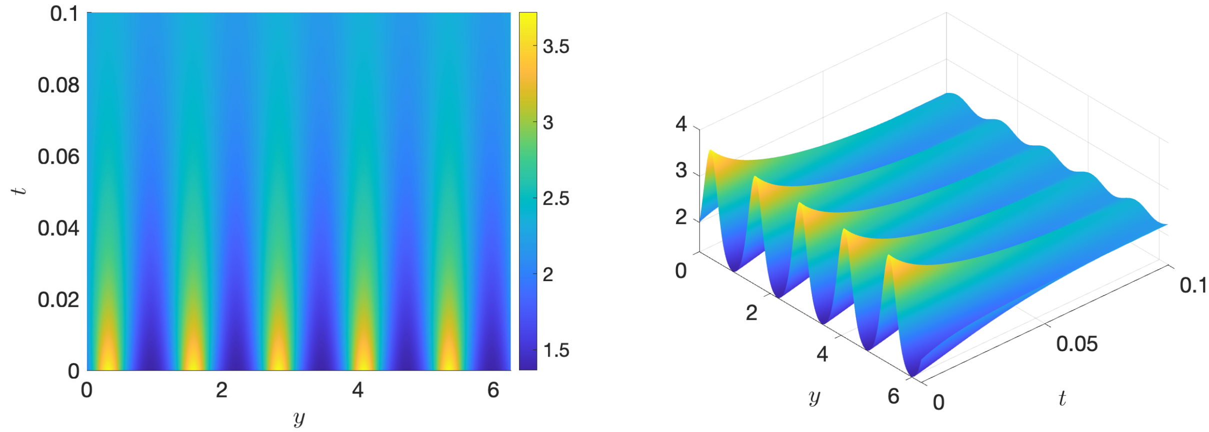


Figure 2: Numerical solution of the linear system (12) representing the initial-boundary value problem for the heat equation (9) . We set for $\alpha = 1$ and $n = 200$ and computed one trajectory corresponding to the initial condition $u_0(y) = 1 + \exp(\sin(5y))$.

Example: Consider the following initial-boundary value problem for the heat equation in the periodic spatial domain $[0, 2\pi]$

$$\begin{cases} \dfrac{\partial u(t,y)}{\partial t} = \alpha \dfrac{\partial^2 u(y,t)}{\partial y^2} & \text{heat equation} \\ u(0,y) = u_0(y) & \text{initial condition} \\ u(t,0) = u(t,2\pi) & \text{periodic boundary conditions} \end{cases} \tag{9}$$

A finite-difference approximation of the PDE (9) on the evenly-spaced grid with n points

$$y_k = (k-1)\Delta y \qquad k = 1, \ldots, n, \qquad \Delta y = \frac{2\pi}{n} \tag{10}$$

yields the n-dimensional *linear dynamical system*[1]

$$\begin{cases} \dfrac{dx_1(t)}{dt} = \dfrac{\alpha}{\Delta y^2}\left[x_2(t) - 2x_1(t) + x_n(t)\right] \\ \dfrac{dx_2(t)}{dt} = \dfrac{\alpha}{\Delta y^2}\left[x_3(t) - 2x_2(t) + x_1(t)\right] \\ \qquad\vdots \\ \dfrac{dx_n(t)}{dt} = \dfrac{\alpha}{\Delta y^2}\left[x_1(t) - 2x_n(t) + x_{n-1}(t)\right] \end{cases} \tag{12}$$

where we defined $x_k(t) = u(y_k, t)$. This system can be written in a vector form as

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}, \tag{13}$$

[1] Recall that the second-order centered finite-difference approximation of the second derivative at y_k is

$$\frac{\partial^2 u(y_k,t)}{\partial y^2} \simeq \frac{u(y_{k+1},t) - 2u(y_k,t) + u(y_{k-1},t)}{\Delta y^2} \tag{11}$$

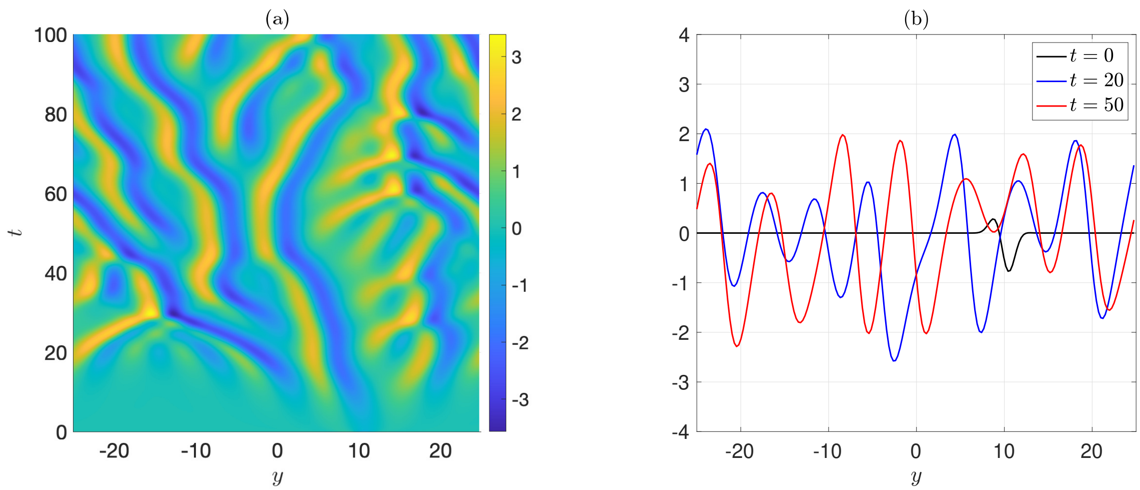


Figure 3: Kuramoto-Sivashinsky equation. Shown is one trajectory of the dynamical system (17) approximating the solution of (15) for $L = 25$ and $n = 200$. In (b) we plot the 200-dimensional solution vector $\boldsymbol{X}(t, \boldsymbol{x}_0)$ at $t = 0$, $t = 20$ and $t = 50$.

where

$$\boldsymbol{A} = \frac{\alpha}{\Delta y^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 1 & \cdots & 0 & 1 & -2 \end{bmatrix} \tag{14}$$

Note that $x_k(t)$ represents an approximation of the solution to the partial differential equation (9) at the grid point $y = y_k$. Hence, by computing the solution to (12), we are computing an approximation of the solution to the PDE (9) at the grid points $(y_1, \ldots, y_n)$.

Example: Consider the following initial-boundary value problem for the Kuramoto-Sivashinsky equation in the periodic spatial domain $[-L, L]$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^4 u}{\partial y^4} = 0 \qquad t \geq 0 \qquad y \in [-L, L] \\ u(y, 0) = \sin(y) e^{-(y-10)^2/2} \\ \text{Periodic boundary conditions} \end{cases} \tag{15}$$

As before, we can approximate the solution to this PDE using finite differences on the spatial grid

$$y_k = -L + (k-1)\Delta y \qquad k = 1, \ldots, n, \qquad \frac{2L}{n}. \tag{16}$$

This yields the n-dimensional nonlinear dynamical system

$$\frac{dx_j}{dt} = \underbrace{-x_j \frac{x_{j+1} - x_{j-1}}{2\Delta y} - \frac{x_{j-1} - 2x_j + x_{j+1}}{\Delta y^2} - \frac{x_{j-2} - 4x_{j-1} - 6x_j + 4x_{j+1} + x_{j-2}}{\Delta y^4}}_{f_j(x_1, \ldots, x_n)} \qquad j = 1, \ldots, n \tag{17}$$

In Figure 3 we show one trajectory of this system for $L = 25$ and $n = 200$ (i.e., a 200-dimensional system) corresponding to the initial condition

$$x_{0k} = u(y_k, 0). \tag{18}$$

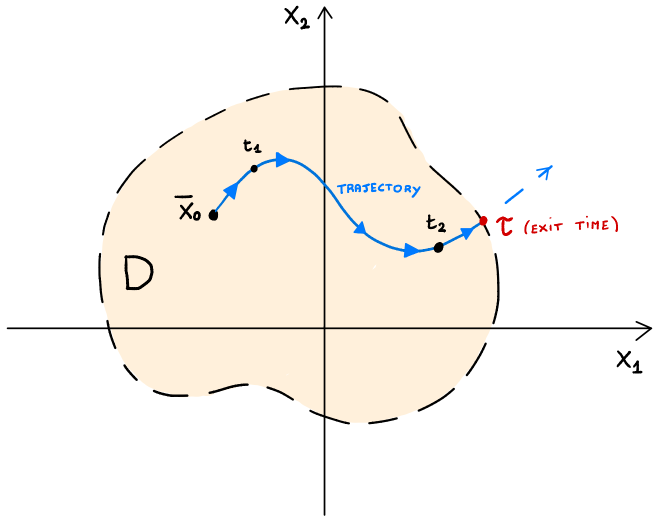


Figure 4: Illustration of the meaning of Theorem 1 in two-dimensions. Shown are the open set $D \subset \mathbb{R}^2$ in which $\boldsymbol{f}(\boldsymbol{x})$ is Lipschitz continuous, the trajectory corresponding to a particular $\boldsymbol{x}_0 \in D$ and the exit time τ for such trajectory.

Well-posedness of the initial value problem

Let us recall the theorem that guarantees existence and uniqueness of the solution to the initial value problem (1).

Theorem 1. Let $D \subset \mathbb{R}^n$ be an open set, $\boldsymbol{x}_0 \in D$. If $\boldsymbol{f} : D \to \mathbb{R}^n$ is Lipschitz continuous in D then there exists a unique solution to the initial value problem (1) within the time interval $[0, \tau[$, where τ is the time instant at which the trajectory $\boldsymbol{x}(t)$ exits[2] the domain D. The solution $\boldsymbol{x}(t)$ is continuously differentiable in $[0, \tau[$.

How do we define Lipschitz continuity for a vector-valued function $\boldsymbol{f}(\boldsymbol{y})$ defined on subset of $\mathbb{R}^n$? By a simple generalization of the definition we gave for one-dimensional functions.

Definition 1. Let D be a subset of $\mathbb{R}^n$, $\boldsymbol{f} : D \to \mathbb{R}^n$. We say that $\boldsymbol{f}$ is Lipschitz continuous in D if there exists a constant $0 \leq L < \infty$ such that

$$\|\boldsymbol{f}(\boldsymbol{x}_1) - \boldsymbol{f}(\boldsymbol{x}_2)\| \leq L \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \quad \text{for all} \quad \boldsymbol{x}_1, \boldsymbol{x}_2 \in D, \tag{19}$$

where $\|\cdot\|$ is any norm defined in $\mathbb{R}^n$ (see Appendix B). Recall, in fact that all norms defined in a finite-dimensional space such as $\mathbb{R}^n$ are *equivalent*.

Similarly to what we have seen for one-dimensional dynamical systems, there conditions that are simpler to verify than Lipschitz continuity.

Lemma 1. Let $\boldsymbol{f}(\boldsymbol{x})$ is of class C^1 in a compact domain $D \subset \mathbb{R}^n$. Then $\boldsymbol{f}(\boldsymbol{x})$ is Lipschitz in D.

The proof of this lemma is provided in Appendix B for the case where D is compact and convex.

Lemma 2. Let $\boldsymbol{f}(\boldsymbol{x})$ be of class C^1 in $D \subseteq \mathbb{R}^n$. If $\boldsymbol{f}(\boldsymbol{x})$ has bounded derivatives $\partial f_i / \partial x_j$ then $\boldsymbol{f}(\boldsymbol{y})$ is Lipschitz continuous in D.

[2] As shown in Figure 4, the "exit time" τ depends on D, $\boldsymbol{f}(x)$ and $\boldsymbol{x}_0$.

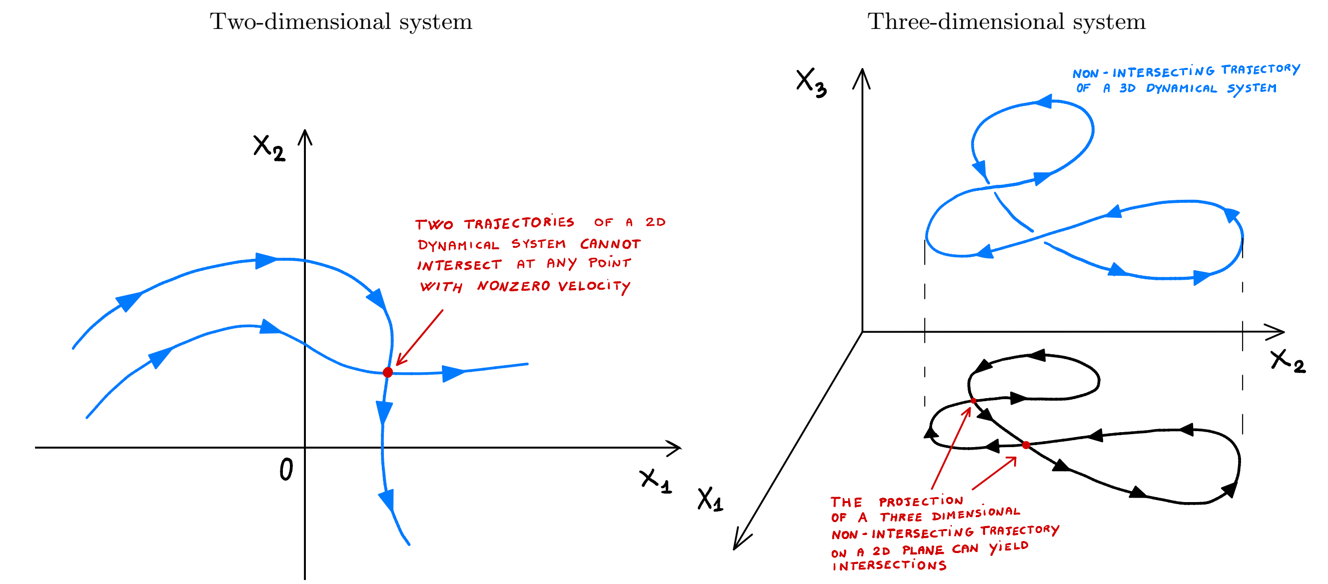


Figure 5: It is impossible for two trajectories to intersect with nonzero velocity any point in phase space. This would make $\boldsymbol{f}(\boldsymbol{x})$ non-unique at such point and also violate the existence and uniqueness Theorem 1.

Global solutions. If $\boldsymbol{f}(\boldsymbol{x})$ is Lipschitz continuous on the entire space $\mathbb{R}^n$ then the solution to the initial value problem (1) is *global*. This means that the solution exists and is unique for all $t \geq 0$. In fact, $x(t)$ never exits the domain in which $\boldsymbol{f}(\boldsymbol{x})$ is Lipschitz continuous, and therefore we can extend τ in Theorem 1 to infinity. It is important to emphasize that existence and uniqueness of the solution to (1) has nothing to do with the smoothness of $\boldsymbol{f}(\boldsymbol{x})$ but rather with the rate at which $\boldsymbol{f}(\boldsymbol{x})$ grows or decays.

Example: The solution to the dynamical systems (8) and (12) is global in time, meaning that it exists and is unique for all $t \geq 0$. In fact, the right hand side of such systems is globally Lipschitz in $\mathbb{R}^2$ and $\mathbb{R}^n$, respectively.

Flow generated by nonlinear dynamical systems

The solution to the initial value problem (1) depends on both the vector field $\boldsymbol{f}(\boldsymbol{x})$ and the initial condition $\boldsymbol{x}_0$. As before, we denote this dependence explicitly by writing the solution as

$$\boldsymbol{x}(t) = \boldsymbol{X}(t, \boldsymbol{x}_0). \tag{20}$$

where $\boldsymbol{X}(t, \boldsymbol{x}_0)$ represents the *flow* generated by (1). Analogous to the one-dimensional case, two solutions corresponding to distinct initial conditions cannot intersect at any finite time t (see Figure 5). If they did, one could use the intersection point as a new initial condition, which would result in multiple solution trajectories originating from the same point, contradicting the existence and uniqueness theorem (Theorem 1). This implies that the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$ is invertible at each finite time[3]. That is, for any given time t, we can uniquely determine the initial condition $\boldsymbol{x}_0$ of the "particle" located at $\boldsymbol{x}(t) = \boldsymbol{X}(t, \boldsymbol{x}_0)$. As a consequence, two particles can never collide at a finite time, nor can a single particle split into multiple trajectories (see Figure 5).

Theorem 2 (Regularity of the flow with respect to $\boldsymbol{x}_0$)**.** Let $D \subset \mathbb{R}^n$ be an open set, $\boldsymbol{x}_0 \in D$. If $\boldsymbol{f} : D \to \mathbb{R}^n$ is Lipschitz continuous in D then the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$ generated by the initial value problem (1), is continuous in $\boldsymbol{x}_0$. Moreover, If $\boldsymbol{f}(\boldsymbol{x})$ is of class $C^k(D)$ (continuously differentiable k-times in D) in D then $\boldsymbol{X}(t, \boldsymbol{x}_0)$ is of class $C^k(D)$ relative to $\boldsymbol{x}_0$ (continuously differentiable k-times with respect to $\boldsymbol{x}_0$).

[3]Solutions may intersect asymptotically as $t \to \infty$, for example when trajectories approach an attracting set.

In summary, Theorem 2 states that the smoother the function $\boldsymbol{f}(\boldsymbol{x})$, the smoother the dependence of the flow $\boldsymbol{X}(t,\boldsymbol{x}_0)$ on the initial condition $\boldsymbol{x}_0$. The n-dimensional mapping $\boldsymbol{X}(t,\boldsymbol{x}_0)$.

Theorem 3 (Regularity of the flow in time)**.** Let $D\subset\mathbb{R}^n$ be an open set, $\boldsymbol{x}_0\in D$. if $\boldsymbol{f}(\boldsymbol{x})$ is of class C^k in D (continuously differentiable k-times in D with continuous derivative), then $\boldsymbol{X}(t,\boldsymbol{x}_0)$ is of class C^{k+1} in time for all $t\in[0,\tau[$, where τ is the time at which $X(t,\boldsymbol{x}_0)$ exits the domain D.

Properties of the flow. The flow $\boldsymbol{X}(t,\boldsymbol{x}_0)$ satisfies the following properties:

- *Identity at time zero:* $\boldsymbol{X}(0,\boldsymbol{x}_0)=\boldsymbol{x}_0$. That is, the flow reduces to the identity mapping at time $t=0$.
- *Invertibility:* The mapping $\boldsymbol{X}(t,\boldsymbol{x}_0)$ is invertible for all times t for which the solution to the initial value problem (1) exists and is unique.
- *Semigroup property:* The flow satisfies the composition rule

$$\boldsymbol{X}(t+s,\boldsymbol{x}_0)=\boldsymbol{X}(t,\boldsymbol{X}(s,\boldsymbol{x}_0))=\boldsymbol{X}(s,\boldsymbol{X}(t,\boldsymbol{x}_0)).$$

  This semigroup property reflects the fact that the solution to the ODE (1) can be restarted at any intermediate time (e.g., t or s) using the state at that time as the new initial condition. This property follows from the existence and uniqueness Theorem 1.
- *Flow equation (forward map):* The flow $\boldsymbol{X}(t,\boldsymbol{x}_0)$ satisfies the following system of first-order partial differential equations:

$$\begin{cases}\dfrac{\partial \boldsymbol{X}(t,\boldsymbol{x}_0)}{\partial t}-\boldsymbol{f}(\boldsymbol{x}_0)\cdot\nabla\boldsymbol{X}(t,\boldsymbol{x}_0)=0,\\ \boldsymbol{X}(0,\boldsymbol{x}_0)=\boldsymbol{x}_0.\end{cases}\tag{21}$$

- *Inverse flow equation:* The inverse flow $\boldsymbol{X}_0(t,\boldsymbol{x})$ satisfies a similar system of first-order PDEs:

$$\begin{cases}\dfrac{\partial \boldsymbol{X}_0(t,\boldsymbol{x})}{\partial t}+\boldsymbol{f}(\boldsymbol{x})\cdot\nabla\boldsymbol{X}_0(t,\boldsymbol{x})=0,\\ \boldsymbol{X}_0(0,\boldsymbol{x})=\boldsymbol{x}.\end{cases}\tag{22}$$

The proofs of these properties are straightforward and closely parallel the corresponding arguments presented for one-dimensional dynamical systems.

Geometric approach

The flow $\boldsymbol{X}(t,\boldsymbol{x}_0)$ maps any initial condition $\boldsymbol{x}_0$ to the solution of the ODE (1) at time t. If we interpret $\boldsymbol{x}_0$ as the initial position of a particle in $\mathbb{R}^n$, then from elementary mechanics it follows that the derivative $d\boldsymbol{X}(0,\boldsymbol{x}_0)/dt=\boldsymbol{f}(\boldsymbol{x}_0)$ represents the velocity of the particle at time $t=0$. The *vector field*[4] $\boldsymbol{f}(\boldsymbol{x})$ associated with the dynamical system indicates the direction in which a particle located at any given point in phase space will move. To illustrate this idea, consider the two-dimensional dynamical system

$$\begin{cases}\dfrac{dx_1}{dt}=f_1(x_1,x_2)\\ \dfrac{dx_2}{dt}=f_2(x_1,x_2)\end{cases}\tag{23}$$

[4]A vector field is a vector-valued function that is continuously indexed by one or more variables. In one-dimensional systems, the vector field $f(x)$ is indexed by the scalar coordinate x and represented as a vector along the line. In two dimensions, the vector field $\boldsymbol{f}(\boldsymbol{x})=(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}))$ has two components, one along x_1 and one along x_2. For three-dimensional systems, the field has three components, and so on.

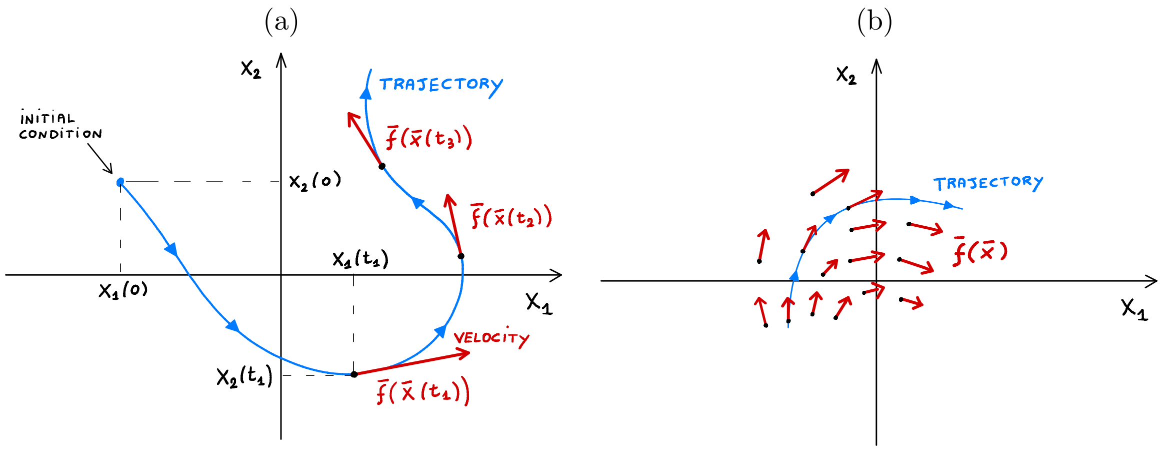


Figure 6: Geometric approach in 2D. (a) Sketch of a trajectory in the phase plane and associated velocity vectors $\boldsymbol{f}(\boldsymbol{x}(t))$. This process can be reversed in the sense that we can also guess how the trajectory $\boldsymbol{x}(t)$ looks like by plotting a bunch of velocity vectors $\boldsymbol{f}(\boldsymbol{x})$ evaluated at different points $\boldsymbol{x}$ in the plane (b).

In Figure 6, we plot the velocity vector

$$\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x})) \tag{24}$$

along a trajectory $\boldsymbol{x}(t)$. This process can be reversed, i.e., by plotting a sufficiently dense set of velocity vectors $\boldsymbol{f}(\boldsymbol{x})$ evaluated at points near a location of interest in phase space, we can qualitatively infer the shape of the trajectory $\boldsymbol{x}(t)$ (see Figure 7(b)).

Example: In Figure 7 we plot the vector fields and corresponding trajectories defined by following two-dimensional dynamical systems

$$\begin{cases} \dot{x}_1 = 2x_1x_2 - 1 \\ \dot{x}_2 = -x_1^2 - x_2^2 + 10 \end{cases} \tag{25}$$

and

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) - \dfrac{1}{10}x_2 \end{cases} \qquad \text{(pendulum with friction)} \tag{26}$$

As we shall see hereafter, the curves $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ are called *nullclines*. Fixed points are at the intersection of nullclines.

Fixed points

If the velocity vector $\boldsymbol{f}(\boldsymbol{x})$ vanishes at some points $\boldsymbol{x}^* \in \mathbb{R}^n$, then a particle placed at that point won't move as time evolves. Such points are called *fixed points* (or *equilibria*) of the dynamical system (1). Mathematically, a fixed point $\boldsymbol{x}^* \in \mathbb{R}^n$ can be defined as

$$\boldsymbol{X}(t, \boldsymbol{x}^*) = \boldsymbol{x}^* \qquad \text{for all } t \geq 0. \tag{27}$$

By differentiating this equation with respect to time we obtain

$$\frac{\partial \boldsymbol{X}(t, \boldsymbol{x}^*)}{\partial t} = \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}^*)) = \boldsymbol{f}(\boldsymbol{x}^*) = 0. \tag{28}$$

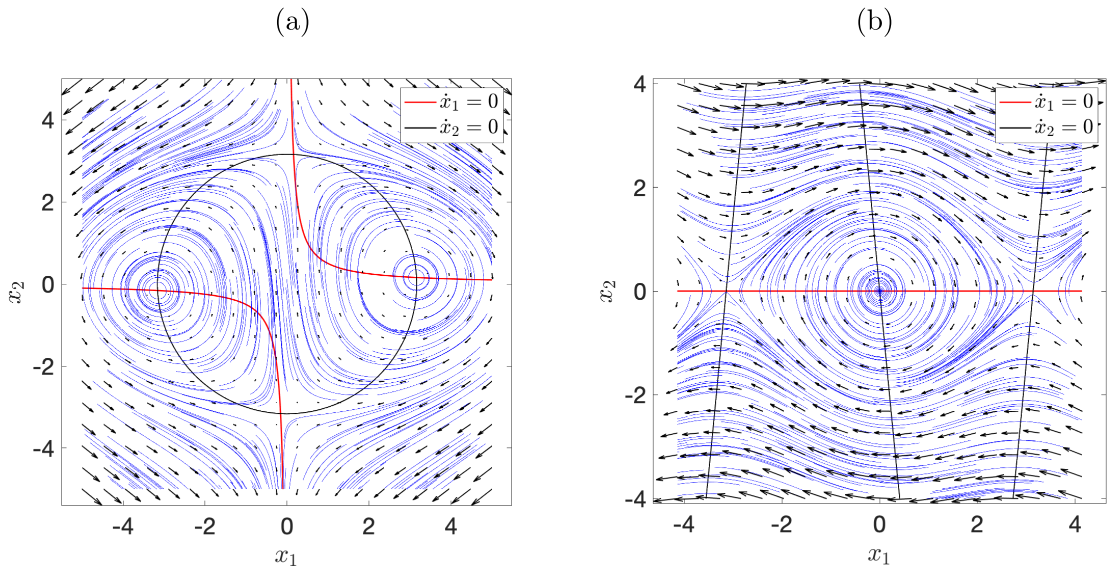


Figure 7: Vector field and trajectories generated by the two-dimensional nonlinear dynamical system (25) (Figure (a)), and (26) (Figure (b)). Shown are also the nullclines for both systems.

Therefore, the fixed points of the system (1) are solutions to the nonlinear system of equations

$$\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}. \tag{29}$$

This system can be written as

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ f_2(x_1, \dots, x_n) = 0 \\ \quad \vdots \\ f_n(x_1, \dots, x_n) = 0 \end{cases} \tag{30}$$

In this form, it is clear that the fixed points (if any) of the system (1) lie at the intersection of the zero level sets[5] of the n functions f_j, each of which depends on n variables. In two-dimensions such zero level sets are identified by the intersection of two surfaces $f_1(x_1, x_2)$ and $f_1(x_1, x_2)$ with the (x_1, x_2) plane

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases} \tag{31}$$

The zero level sets of $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are called **nullclines**. This terminology arises from the fact that the vector field $\boldsymbol{f}(x_1, x_2)$ is vertical at every point on the nullcline defined by $f_1(x_1, x_2) = 0$, and horizontal at every point on the nullcline defined by $f_2(x_1, x_2) = 0$. Consequently, trajectories intersect the nullclines $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ vertically and horizontally, respectively (see, for example, Figure 7 or Figure 1).

Example: Let us calculate the nullclines of the dynamical system (25). The first nullcline is

$$f_1(x_1, x_2) = 2x_1x_2 - 1 = 0 \qquad \Rightarrow \qquad x_2 = \frac{1}{2x_1} \qquad (\text{nullcline } \dot{x}_1 = 0), \tag{32}$$

[5]The fixed points can be computed analytically only for simple prototype dynamical systems. In general, finding fixed points requires a numerical root-finding algorithm for nonlinear systems of algebraic equations, such as Newton's method.

i.e., the hyperbola depicted in red in Figure 7(a). The second nullcline is a circle with radius $\sqrt{10}$ centered at the origin

$$f_2(x_1, x_2) = -x_1^2 - x_2^2 + 10 = 0 \qquad \Rightarrow \qquad x_1^2 + x_2^2 = 10 \qquad (\text{nullcline } \dot{x}_2 = 0). \tag{33}$$

This is depicted in black in Figure 7(a). We also see that the trajectories of the system intersect the nullcline $\dot{x}_2 = 0$ (black curve) horizontally. In fact, such nullcline represents the set of points in the phase plane where the velocity has zero vertical component. Similarly, the trajectories intersect the nullcline $\dot{x}_1 = 0$ (red curve) vertically. The four fixed points of the system are at the intersection of the nullclines and can be computed analytically.

Example: Let us calculate the nullclines and the fixed points of the dynamical system (26). The first nullcline is

$$f_1(x_1, x_2) = x_2 = 0 \qquad \Rightarrow \qquad x_2 = 0 \qquad (\text{nullcline } \dot{x}_1 = 0), \tag{34}$$

Such nullcline is plotted in red in Figure 7(b). The second nullcline is

$$f_2(x_1, x_2) = \sin(x_1) - x_2/10 = 0 \qquad \Rightarrow \qquad x_2 = 10\sin(x_1) \qquad (\text{nullcline } \dot{x}_2 = 0), \tag{35}$$

and it is plotted in black in Figure 7(b). The fixed points are at the intersection of the nullclines. In this case we obtain two physically different fixed points:

$$x_1^* = (0, 0) \qquad x_2^* = (\pi, 0) \tag{36}$$

corresponding to a pendulum in a vertical position, i.e., $x_1 = 0$ or $x_1 = \pi$ with zero velocity $x_2 = 0$.

Stability analysis of fixed points

A quick look at the phase portraits in Figure 7 suggests that the dynamics in a neighborhood of a fixed point can be quite different. Such dynamics can often be computed via a linearization process that is similar to the process we used in one-dimensional dynamical systems. The idea is "zoom-in" on a fixed point $\boldsymbol{x}^*$ and compute the orbits of the dynamical systems in a small neighborhood of $\boldsymbol{x}^*$ by solving a linearized version of the system (1). To this end, let us first define what we mean by stability of a fixed point.

Definition 2. Let $\boldsymbol{f}(\boldsymbol{x})$ be a locally Lipschitz vector field defined over a domain $D \subseteq \mathbb{R}^n$. Let $\boldsymbol{x}^*$ be a fixed point, i.e., $\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$. We say that $\boldsymbol{x}^*$ is *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ (dependent on ϵ) such that

$$\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| < \delta \quad \Rightarrow \quad \|\boldsymbol{X}(t, \boldsymbol{x}_0) - \boldsymbol{x}^*\| < \epsilon \qquad \forall t \geq 0, \tag{37}$$

We say that $\boldsymbol{x}^*$ is *asymptotically stable* if for all $\epsilon > 0$ there exists $\delta > 0$ (dependent on ϵ) such that

$$\|\boldsymbol{x}_0 - \boldsymbol{x}^*\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} \|\boldsymbol{X}(t, \boldsymbol{x}_0) - \boldsymbol{x}^*\| = 0. \tag{38}$$

Next, consider an initial condition $\boldsymbol{x}_0$ that lies very close to a fixed point $\boldsymbol{x}^*$, and define the perturbation

$$\boldsymbol{\eta}(t, \boldsymbol{x}_0) = \boldsymbol{X}(t, \boldsymbol{x}_0) - \boldsymbol{x}^*. \tag{39}$$

Expanding the function $\boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) = \boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\eta}(t, \boldsymbol{x}_0))$ in a neighborhood of $\boldsymbol{x}^*$, i.e., for small $\boldsymbol{\eta}(t, \boldsymbol{x}_0)$ yields

$$\boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\eta}(t, \boldsymbol{x}_0)) = \underbrace{\boldsymbol{f}(\boldsymbol{x}^*)}_{=\boldsymbol{0}} + \boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{x}^*)\boldsymbol{\eta}(t, \boldsymbol{x}_0) + \cdots, \tag{40}$$

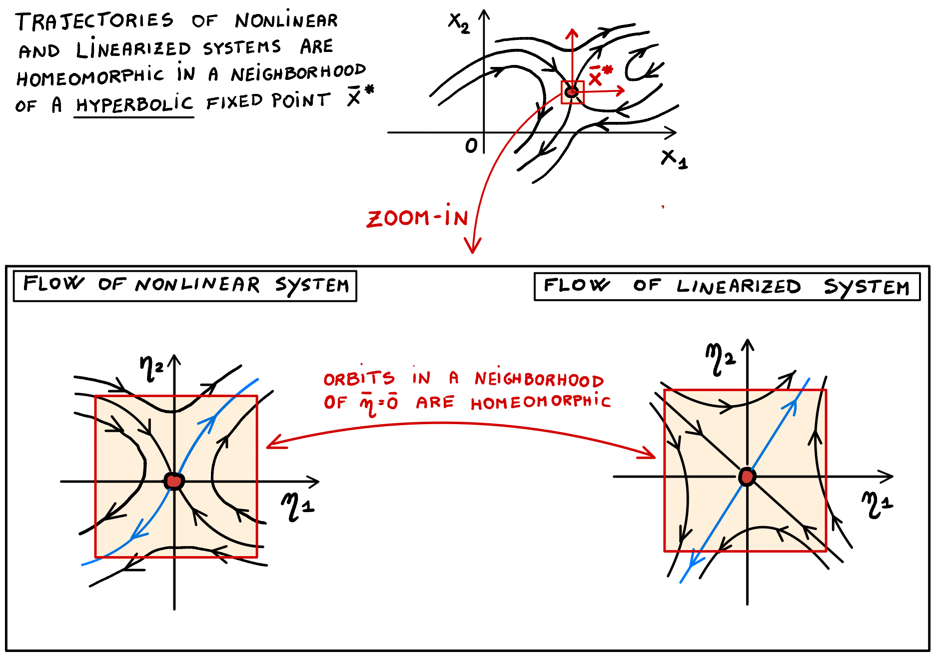


Figure 8: Geometric interpretation of the Hartman–Grobman Theorem 4. The trajectories of a nonlinear dynamical system in a neighborhood of a hyperbolic fixed point are *homeomorphic* to those of the linearized system at $\boldsymbol{x}^*$. This means that while the trajectories of the nonlinear and linearized systems are not identical, they can be related through a continuous transformation with a continuous inverse. In other words, the qualitative structure of the phase portrait near $\boldsymbol{x}^*$ is preserved under this homeomorphism.

where

$$\boldsymbol{J_f}(\boldsymbol{x}^*) = \begin{bmatrix} \dfrac{\partial f_1(\boldsymbol{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial f_1(\boldsymbol{x}^*)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n(\boldsymbol{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial f_n(\boldsymbol{x}^*)}{\partial x_n} \end{bmatrix} \tag{41}$$

denotes the Jacobian matrix[6] of $\boldsymbol{f}$ evaluated at $\boldsymbol{x}^*$. Hence, the first-order approximation of the nonlinear dynamical system (1) at $\boldsymbol{x}^*$ can be written as

$$\begin{cases} \dfrac{d\boldsymbol{\eta}}{dt} = \boldsymbol{J_f}(\boldsymbol{x}^*)\boldsymbol{\eta} \\ \boldsymbol{\eta}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0 - \boldsymbol{x}^* \end{cases} \tag{42}$$

Theorem 4 (Hartman-Grobman)**.** Let $\boldsymbol{x}^* \in \mathbb{R}^n$ be a fixed point of the dynamical system (1). If the Jacobian (41) has no eigenvalue with zero real part then there exists a homeomorphism (i.e., continuous invertible mapping with continuous inverse) defined on some neighborhood U of $\boldsymbol{x}^*$ that takes orbits of the system (1) and maps them into orbits of the linearized system (39)-(42). The mapping preserves the orientation of the orbits.

An outline of the proof is given in L. Perko, *Differential equations and dynamical systems*, page 121.

[6] The Jacobian of $\boldsymbol{f}(\boldsymbol{x})$ is a matrix-valued function that maps the vector field $\boldsymbol{f}(\boldsymbol{x})$ to an $n \times n$ matrix whose entries are functions of $\boldsymbol{x}$. When the Jacobian is evaluated at a particular point $\boldsymbol{x}^*$, it becomes a matrix with real-valued entries (assuming $\boldsymbol{f}$ is a real-valued function).

Remark: Theorem (4) states that if $\boldsymbol{x}^*$ is a *hyperbolic*[7] fixed point, then the flow of the nonlinear system (1) in a neighborhood $U \subset \mathbb{R}^n$ of $\boldsymbol{x}^*$ is *homeomorphic* to the flow of the corresponding linearized system (42). That is, the trajectories of the nonlinear and linear systems can be mapped to each other by a continuous bijection

$$\boldsymbol{h} : U \mapsto V, \tag{43}$$

with a continuous inverse, where V is the image of U under $\boldsymbol{h}$. Stated mathematically, the theorem asserts that there exists a homeomorphism $\boldsymbol{h}$ such that

$$\boldsymbol{h}(\boldsymbol{X}(t,\boldsymbol{x}_0) - \boldsymbol{x}^*) = e^{t\boldsymbol{J_f}(\boldsymbol{x}^*)}\boldsymbol{h}(\boldsymbol{x}_0 - \boldsymbol{x}^*), \qquad \text{i.e.} \qquad \boldsymbol{X}(t,\boldsymbol{x}_0) = \boldsymbol{x}^* + \boldsymbol{h}^{-1}\left(e^{t\boldsymbol{J_f}(\boldsymbol{x}^*)}\boldsymbol{h}(\boldsymbol{x}_0 - \boldsymbol{x}^*)\right), \tag{44}$$

Questions: At this point, it is natural to ask the following questions:

1. To analyze the stability of hyperbolic fixed points, we must compute the flow of the linear system (42) that approximates (2) at $\boldsymbol{x}^*$. Is there a general method for computing such flows? The answer is yes. **Flows of linear systems** are important in their own right, as many systems, e.g., the discretized PDE system (12), are inherently linear.

2. What happens if the fixed point is non-hyperbolic? In such cases, as we will see, one could use a generalization of the Hartman-Grobman theorem known as **center manifold theorem**.

3. Is there an alternative method to study the stability of fixed points that does not rely on computing trajectories of the linearized system or analyzing center manifolds? In principle, yes. Such a method was developed by Lyapunov in 1892 and is known as **Lyapunov stability theory**.

Appendix A: Equivalent norms in $\mathbb{R}^n$

As is well known, all norms defined in a finite-dimensional vector space such as $\mathbb{R}^n$ are *equivalent*. This means that if we pick two arbitrary norms in $\mathbb{R}^n$, say $\|\cdot\|_a$ and $\|\cdot\|_b$, then there exist two numbers C_1 and C_2 such that

$$C_1 \|\boldsymbol{x}\|_a \leq \|\boldsymbol{x}\|_b \leq C_2 \|\boldsymbol{x}\|_a \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^n. \tag{45}$$

The most common norms in $\mathbb{R}^n$ are

$$\|\boldsymbol{x}\|_\infty = \max_{k=1,..,n} |x_k|, \tag{46}$$

$$\|\boldsymbol{x}\|_1 = \sum_{k=1}^n |x_k|, \tag{47}$$

$$\|\boldsymbol{x}\|_2 = \left(\sum_{k=1}^n |x_k|^2\right)^{1/2}, \tag{48}$$

$$\vdots \tag{49}$$

$$\|\boldsymbol{x}\|_p = \left(\sum_{k=1}^n |y_k|^p\right)^{1/p} \qquad p \in \mathbb{N}\setminus\{\infty\}. \tag{50}$$

Based on these definitions it can be shown that, e.g., that

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_1 \leq n\|\boldsymbol{x}\|_\infty, \tag{51}$$

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2, \tag{52}$$

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \sqrt{n}\|\boldsymbol{x}\|_\infty. \tag{53}$$

[7] A fixed point $\boldsymbol{x}^*$ is called *hyperbolic* if the Jacobian matrix $\boldsymbol{J_f}(\boldsymbol{x}^*)$ has no eigenvalues with zero real part.

Therefore if the $\boldsymbol{f}(\boldsymbol{x})$ is Lipschitz continuous in D with respect to the 1-norm, i.e.,

$$\|\boldsymbol{f}(\boldsymbol{x}_1)-\boldsymbol{f}(\boldsymbol{x}_2)\|_1 \leq L_1\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_1 \quad \text{for all} \quad \boldsymbol{x}_1,\boldsymbol{x}_2\in D \tag{54}$$

then it is also Lipschitz continuous in D with respect to the uniform norm. In fact, by using (51) we have

$$\|\boldsymbol{f}(\boldsymbol{x}_1)-\boldsymbol{f}(\boldsymbol{x}_2)\|_\infty \leq \underbrace{L_1 n}_{L_\infty}\|\boldsymbol{x}_1-\boldsymbol{x}_2\|_\infty. \tag{55}$$

Similarly, by using (52), we see that if $\boldsymbol{f}$ is Lipschitz in D with respect to the 1-norm then $\boldsymbol{f}$ is Lipschitz in D with respect to the 2-norm.

Appendix B: Proof of Lemma 1

Let $D\subseteq\mathbb{R}^n$ be a compact convex domain and let

$$M=\max_{\boldsymbol{x}\in D}\left|\frac{\partial f_j(\boldsymbol{x})}{\partial x_i}\right|. \tag{56}$$

Clearly M exists and is finite because we assumed that D is compact and that $\boldsymbol{f}$ is of class C^1 in D[8]. Consider two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in D, and the line that connects $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$, i.e.,

$$\boldsymbol{z}(s)=(1-s)\boldsymbol{x}_1+s\boldsymbol{x}_2 \qquad s\in[0,1]. \tag{57}$$

Since D is convex, we have that the line $\boldsymbol{z}(s)$ lies entirely within D. Therefore we can use the mean value theorem applied to the one-dimensional function $f_i(\boldsymbol{z}(s))$ ($s\in[0,1]$) to obtain

$$f_i(\boldsymbol{x}_2)-f_i(\boldsymbol{x}_1)=\nabla f_i(\boldsymbol{z}(s^*),t)\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1) \qquad \text{for some } s^*\in[0,1]. \tag{58}$$

By taking the absolute value and using the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
|f_i(\boldsymbol{x}_1)-f_i(\boldsymbol{x}_1)|^2 =&\left|\sum_{j=1}^n \frac{\partial f_i(\boldsymbol{z}(s^*))}{\partial x_j}(x_{2j}-x_{1j})\right|^2\\
\leq&\left|\sum_{j=1}^n \frac{\partial f_i(\boldsymbol{z}(s^*))}{\partial x_j}\right|^2\left|\sum_{j=1}^n (x_{2j}-x_{1j})\right|^2\\
\leq& nM^2\|\boldsymbol{x}_2-\boldsymbol{x}_1\|_2^2.
\end{aligned} \tag{59}$$

This implies that

$$\|\boldsymbol{f}(\boldsymbol{x}_2)-\boldsymbol{f}(\boldsymbol{x}_1)\|_2 \leq \underbrace{nM}_{L_2}\|\boldsymbol{x}_2-\boldsymbol{x}_1\|_2. \tag{60}$$

i.e., $\boldsymbol{f}(\boldsymbol{y},t)$ is Lipschitz continuous in the 2-norm, or any other norm that is equivalent to the 2-norm. In particular, by using the inequalities (51)-(53) we have that $\boldsymbol{f}(\boldsymbol{x})$ is Lipschitz continuous relative to the 1-norm.

[8] A compact domain is by definition bounded and closed. The minimum and maximum of a continuous function in defined on a compact domain is attained at some points within the domain or on its boundary. Note that this is not true if the domain is not compact. For example, the function $f(y)=1/y$ is continuously differentiable on $]0,1]$ (bounded domain by not compact), but the function is unbounded on $]0,1]$.

Appendix C: Explicit midpoint method

We can re-write the Cauchy problem (1) as an integral equation

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \boldsymbol{f}(\boldsymbol{x}(s))ds. \tag{61}$$

This form is quite handy to derive numerical methods to solve (1) based on quadrature rules applied to the one-dimensional integral at the right hand side. For instance, consider a partition of the $[0, T]$ into an evenly-spaced grid points such that $t_{i+1} = t_i + \Delta t$, and write (61) within each time interval

$$\boldsymbol{x}(t_{i+1}) = \boldsymbol{x}(t_i) + \int_{t_i}^{t_{i+1}} \boldsymbol{f}(\boldsymbol{x}(s))ds. \tag{62}$$

Explicit midpoint method. By approximating the integral at the right hand side of (62), e.g., using the midpoint rule yields

$$\int_{t_i}^{t_{i+1}} \boldsymbol{f}(\boldsymbol{x}(s))ds \simeq \Delta t \boldsymbol{f}\left(\boldsymbol{x}\left(t_i + \frac{\Delta t}{2}\right)\right) \tag{63}$$

At this point, we can approximate $\boldsymbol{x}(t_i + \Delta t/2)$ using the Euler forward method

$$\boldsymbol{x}\left(t_i + \frac{\Delta t}{2}\right) \simeq \boldsymbol{x}(t_i) + \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{x}(t_i)) \tag{64}$$

to obtain the *explicit midpoint method*

$$\boldsymbol{x}(t_{i+1}) = \boldsymbol{x}(t_i) + \Delta t \boldsymbol{f}\left(\boldsymbol{x}(t_i) + \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{x}(t_i))\right). \tag{65}$$

The explicit midpoint method is a one-step method that belongs to the larger class of Runge-Kutta methods.