Linear dynamical systems

Consider the following n-dimensional linear dynamical system

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t) \cdots x_n(t)]^T$ is a column vector of phase variables, and $\boldsymbol{A} \in M_{n\times n}(\mathbb{R})$ is a $n \times n$ matrix with real coefficients. It is straightforward to verify that the (linear) function $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ is Lipschitz continuous on $\mathbb{R}^n$. Indeed, for any matrix norm $\|\boldsymbol{A}\|$ compatible with the vector norm $\|\boldsymbol{x}\|$ (see Appendix C), we have

$$\|\boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{A}\boldsymbol{x}_2\| = \|\boldsymbol{A}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\| \leq \|\boldsymbol{A}\|\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|, \tag{2}$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$. Alternatively, note that the function $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ has bounded partial derivatives everywhere in $\mathbb{R}^n$, provided the entries of $\boldsymbol{A}$ are finite:

$$\frac{\partial f_i(\boldsymbol{x})}{\partial x_j} = A_{ij} < \infty, \qquad \text{for all } i, j = 1, \ldots, n. \tag{3}$$

Therefore, by Lemma 2 from course note 2, it follows immediately that the solution of equation (1) is global, i.e., it exists and is unique for all $t \geq 0$. Furthermore, since $\boldsymbol{A}\boldsymbol{x}$ is infinitely differentiable on $\mathbb{R}^n$, Theorems 2 and 3 from course note 2 imply that the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$ generated by the linear dynamical system (1) is of class C^∞ with respect to both t and $\boldsymbol{x}_0$.

Fixed points and their stability properties

Fixed points of the linear dynamical system (1) are solutions of the linear equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}_{\mathbb{R}^n}, \tag{4}$$

i.e., they lie at the intersection of n hyper-planes passing through the origin in $\mathbb{R}^n$. Such hyper-planes are defined by the linear equations

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = 0 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = 0 \\ \quad\vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n = 0 \end{cases} \tag{5}$$

Clearly, if the matrix $\boldsymbol{A}$ is invertbile then we have a unique fixed point at

$$\boldsymbol{x}^* = \boldsymbol{0}_{\mathbb{R}^n}. \tag{6}$$

On the other hand, if the matrix $\boldsymbol{A}$ is not invertible then we have an infinite number of fixed points, i.e., all points in the nullspace[1] of $\boldsymbol{A}$ are fixed points.

Example: The fixed points of the 2D linear dynamical system defined by the rank 1 matrix

$$\boldsymbol{A} = \begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \tag{7}$$

[1]Recall that the nullspace of a matrix $\boldsymbol{A}$ is the set of vectors that are sent to the zero vector by applying $\boldsymbol{A}$. The nullspace of an $n \times n$ matrix is a vector subspace of $\mathbb{R}^n$.

are obtained by solving

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad x_2 = 2x_1 \tag{8}$$

Hence, in this case we have an infinite number of fixed points sitting on a line with slope 2 passing through the origin of the phase plane (x_1, x_2).

Stability analysis. The stability of fixed points (either one or infinite) is determined by the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of the matrix $\boldsymbol{A}$. If $\boldsymbol{A}$ is invertible (no zero-eigenvlaue) we only have one fixed point (the origin) and

$$\begin{aligned} \text{Re}(\lambda_i) \leq 0 \quad &\Rightarrow \quad \text{fixed points are } \textit{stable} \\ \text{Re}(\lambda_i) < 0 \quad &\Rightarrow \quad \text{fixed points are } \textit{asymptotically stable} \\ \text{Re}(\lambda_i) > 0 \quad &\Rightarrow \quad \text{fixed points are } \textit{unstable} \end{aligned}$$

Flow generated by linear dynamical systems

As shown in Appendix D, the analytical solution of the initial value problem (1) can be formally expressed in terms of a matrix exponential[2], i.e.,

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = e^{t\boldsymbol{A}}\boldsymbol{x}_0. \tag{10}$$

This expression shows that the flow is indeed of class C^∞, in both $\boldsymbol{x}_0$ and t, as anticipated above. Hereafter we take a linear algebraic approach to the problem of solving the linear system of ODEs (1), i.e., we focus on linear algebraic techniques to compute the matrix exponential $e^{t\boldsymbol{A}}$ explicitly in terms of the spectral properties (eigenvalues, eigenvectors and generalized eigenvectors) of the matrix $\boldsymbol{A}$.

Computation of the matrix exponential. The matrix exponential in appearing in (10) can be written explicitly in terms of the eigenvalues and the eigenvectors (or generalized eigenvectors) of the matrix $\boldsymbol{A}$. In Appendix A and Appendix B we provide a thorough review of the matrix eigenvalue problem, including calculation of the eigenvalues, eigenvectors and generalized eigenvectors of a matrix. Please read through Appendix A and Appendix B very carefully, as everything that is discussed hereafter assumes that you are familiar with eigenvalues, eigenspaces, generalized eigenvectors, and similarity transformations. The computation of the matrix the matrix exponential $e^{t\boldsymbol{A}}$, and therefore the solution (10) of the linear system (1), differs depending on whether or not

- the matrix $\boldsymbol{A}$ is diagonalizable,
- the matrix $\boldsymbol{A}$ is not diagonalizable.

For definitions of diagonalizable and non-diagonalizable matrices, see Appendix A. As we will see, the non-diagonalizable case includes the diagonalizable one. Therefore, in principle, it would be sufficient to develop the formula for the matrix exponential in the case where $\boldsymbol{A}$ is non-diagonalizable. However, for clarity of exposition, we present the two cases separately.

[2]Recall that the matrix exponential is formally defined by the power series

$$e^{t\boldsymbol{A}} = I + t\boldsymbol{A} + \frac{t^2}{2}\boldsymbol{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k \boldsymbol{A}^k}{k!}, \tag{9}$$

which converges uniformly for all $t \geq 0$.

Matrix exponential for diagonalizable matrices. If $\boldsymbol{A}$ is diagonalizable then there exists a set of n distinct eigenvectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ and a similarity transformation $\boldsymbol{P}$ such that (see Appendix A)

$$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Lambda}, \tag{11}$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \tag{12}$$

is a diagonal matrix that contains all eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $\boldsymbol{A}$ and

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \tag{13}$$

is a matrix that contains all eigenvectors of $\boldsymbol{A}$. Each vector $\boldsymbol{v}_i$ in (13) is a column vector. Since the matrix $\boldsymbol{P}$ is invertible we have

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}. \tag{14}$$

This matrix factorization is very effective when computing the matrix powers appearing in the definition of the matrix exponential (9). In fact,

$$\boldsymbol{A}^2 = \boldsymbol{P}\boldsymbol{\Lambda}\underbrace{\boldsymbol{P}^{-1}\boldsymbol{P}}_{\boldsymbol{I}}\boldsymbol{\Lambda}\boldsymbol{P}^{-1} = \boldsymbol{P}\boldsymbol{\Lambda}^2\boldsymbol{P}^{-1}. \tag{15}$$

Similarly,

$$\boldsymbol{A}^3 = \boldsymbol{P}\boldsymbol{\Lambda}^3\boldsymbol{P}^{-1}, \qquad \cdots, \boldsymbol{A}^k = \boldsymbol{P}\boldsymbol{\Lambda}^k\boldsymbol{P}^{-1}. \tag{16}$$

This implies that

$$e^{t\boldsymbol{A}} = \boldsymbol{P}\left(I + t\boldsymbol{\Lambda} + \frac{t^2}{2}\boldsymbol{\Lambda}^2 + \cdots\right)\boldsymbol{P}^{-1} = \boldsymbol{P}e^{t\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}. \tag{17}$$

The exponential the diagonal matrix $\boldsymbol{\Lambda}$ in (12) is easily obtained as

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} e^{t\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{t\lambda_n} \end{bmatrix}. \tag{18}$$

Hence, when $\boldsymbol{A}$ is diagonalizable, the analytical solution to equation (1) can be computed through the following steps:

1. Compute the eigenvalues and the eigenvectors of $\boldsymbol{A}$;
2. Construct the matrix $\boldsymbol{P}$ in (13) and the matrix exponential (18);
3. Compute the analytical solution of (1) using matrix-vector products

$$\boxed{\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{P}e^{t\boldsymbol{\Lambda}}\boldsymbol{P}^{-1}\boldsymbol{x}_0.} \tag{19}$$

Matrix exponential for non-diagonalizable matrices. If the matrix $\boldsymbol{A}$ is not diagonalizable then there exist a similarity transformation $\boldsymbol{P}$ such that

$$\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{J}, \tag{20}$$

where (assuming that $\boldsymbol{A}$ has p distinct eigenvalues[3])

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{J}_p \end{bmatrix} \tag{21}$$

is a block-diagonal matrix called the Jordan form of $\boldsymbol{A}$ (see Appendix B and Table 1). The matrix

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \tag{22}$$

is the matrix that contains the eigenvectors and the generalized eigenvectors of $\boldsymbol{A}$ columnwise.

Since the matrix $\boldsymbol{P}$ is invertible (eigenvectors and generalized eigenvectors are linearly independent) we have the matrix factorization

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{J}\boldsymbol{P}^{-1}, \tag{23}$$

By following exactly the same steps as in (15)-(17) we obtain the following expression for the matrix exponential of $\boldsymbol{A}$ in the case where $\boldsymbol{A}$ is non-diagonalizable

$$e^{t\boldsymbol{A}} = \boldsymbol{P}e^{t\boldsymbol{J}}\boldsymbol{P}^{-1}. \tag{24}$$

The Jordan canonical form of $\boldsymbol{A}$ is a block-diagonal matrix (see equation (159)), with blocks given in Table 1. The matrix exponential of a block-diagonal matrix is a matrix that has the exponential of each block in the diagonal

$$e^{t\boldsymbol{J}} = \begin{bmatrix} e^{t\boldsymbol{J}_1} & & & \\ & e^{t\boldsymbol{J}_2} & & \\ & & \ddots & \\ & & & e^{t\boldsymbol{J}_p} \end{bmatrix}. \tag{25}$$

In Table 1 we summarize the Jordan blocks corresponding to different types of eigenvalues. The mathematical proof of each Jordan block is given in Appendix B.

Hence, when $\boldsymbol{A}$ is not diagonalizable, the analytical solution to equation (1) can be computed through the following steps:

1. Compute the eigenvalues, the eigenvectors, and the generalized eigenvectors of $\boldsymbol{A}$;
2. Construct the the matrix $\boldsymbol{J}$ using the Jordan blocks in Table 1;
3. Construct the matrix $\boldsymbol{P}$ in (22) and the matrix exponential (25) by exponentiating each Jordan block as in Table 1;
4. Compute the analytical solution of (1) using matrix-vector products

$$\boxed{\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{P}e^{t\boldsymbol{J}}\boldsymbol{P}^{-1}\boldsymbol{x}_0.} \tag{26}$$

Fundamental matrix. In the theory of autonomous linear ODEs the general solution of the system (1) is often expressed in terms of a *fundamental matrix* $\boldsymbol{\Phi}(t)$ as

$$\boldsymbol{x}_g(t) = \boldsymbol{\Phi}(t)\boldsymbol{c}, \tag{27}$$

where $\boldsymbol{c}$ is an arbitrary vector. Enforcing the initial condition $\boldsymbol{x}_g(0) = \boldsymbol{x}_0$ we find that

$$\boldsymbol{c} = \boldsymbol{\Phi}^{-1}(0)\boldsymbol{x}_0. \tag{28}$$

[3]The sum of the algebraic multiplicities of the eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$ must be equal to n.

Properties of the eigenvalue	Jordan block	Exponential of Jordan block
λ_i has algebraic multiplicity one	$\boldsymbol{J}_i = \begin{bmatrix}\lambda_i\end{bmatrix}$	$e^{t\boldsymbol{J}_i} = \begin{bmatrix}e^{t\lambda_i}\end{bmatrix}$
λ_i has algebraic multiplicity two and geometric multiplicity two	$\boldsymbol{J}_i = \begin{bmatrix}\lambda_i & 0\\ 0 & \lambda_i\end{bmatrix}$	$e^{t\boldsymbol{J}_i} = \begin{bmatrix}e^{t\lambda_i} & 0\\ 0 & e^{t\lambda_i}\end{bmatrix}$
λ_i has algebraic multiplicity two and geometric multiplicity one	$\boldsymbol{J}_i = \begin{bmatrix}\lambda_i & 1\\ 0 & \lambda_i\end{bmatrix}$	$e^{t\boldsymbol{J}_i} = \begin{bmatrix}e^{t\lambda_i} & te^{t\lambda_i}\\ 0 & e^{t\lambda_i}\end{bmatrix}$
λ_i has algebraic multiplicity three and geometric multiplicity three	$\boldsymbol{J}_i = \begin{bmatrix}\lambda_i & 0 & 0\\ 0 & \lambda_i & 0\\ 0 & 0 & \lambda_i\end{bmatrix}$	$e^{t\boldsymbol{J}_i} = \begin{bmatrix}e^{t\lambda_i} & 0 & 0\\ 0 & e^{t\lambda_i} & 0\\ 0 & 0 & e^{t\lambda_i}\end{bmatrix}$
λ_i has algebraic multiplicity three and geometric multiplicity two	$\boldsymbol{J}_i = \begin{bmatrix}\lambda_i & 0 & 0\\ 0 & \lambda_i & 1\\ 0 & 0 & \lambda_i\end{bmatrix}$	$e^{t\boldsymbol{J}_i} = \begin{bmatrix}e^{t\lambda_i} & 0 & 0\\ 0 & e^{t\lambda_i} & te^{t\lambda_i}\\ 0 & 0 & e^{t\lambda_i}\end{bmatrix}$
λ_i has algebraic multiplicity three and geometric multiplicity one	$\boldsymbol{J}_i = \begin{bmatrix}\lambda_i & 1 & 0\\ 0 & \lambda_i & 1\\ 0 & 0 & \lambda_i\end{bmatrix}$	$e^{t\boldsymbol{J}_i} = \begin{bmatrix}e^{t\lambda_i} & te^{t\lambda_i} & t^2e^{t\lambda_i}/2\\ 0 & e^{t\lambda_i} & te^{t\lambda_i}\\ 0 & 0 & e^{t\lambda_i}\end{bmatrix}$

Table 1: Jordan blocks and matrix exponentials of Jordan blocks (see Appendix B) corresponding to eigenvalues λ_i with different algebraic and geometric multiplicities.

Substituting this expression for $\boldsymbol{c}$ back into (27) gives

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0)\boldsymbol{x}_0. \tag{29}$$

Comparing this expression to (10) suggests that we can equivalently write the matrix exponential of $\boldsymbol{A}$ as

$$e^{t\boldsymbol{A}} = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(0). \tag{30}$$

Regarding the analytical expression of the fundamental matrix $\boldsymbol{\Phi}(t)$, it can be obtained by comparing (30) with (24). This yields

$$\boldsymbol{\Phi}(t) = \boldsymbol{P}e^{t\boldsymbol{J}} \tag{31}$$

where $\boldsymbol{P}$ is the matrix (22) that has the eigenvectors and generalized eigenvectors of $\boldsymbol{A}$ as columns. As before, the exponential of the Jordan canonical form of $\boldsymbol{A}$, i.e., $e^{t\boldsymbol{J}}$, can be computed by using (25) and exponentiating each Jordan block as in Table 1.

Two-dimensional linear dynamical systems

In this section we compute the analytical solution/flow of several prototype two-dimensional dynamical systems using the mathematical techniques we just discussed. Specifically, we study the flow corresponding to the *saddle node*, *spiral*, *center*, and *degenerate node*.

Saddle node. Consider the linear dynamical system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}2 & 3\\ 3 & -6\end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{32}$$

We have seen in Appendix A (Example 4) that the eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = 3, \qquad \lambda_2 = -7. \tag{33}$$

Since the eigenvalues are simple, the matrix $\boldsymbol{A}$ is diagonalizable. A basis for the eigenspace corresponding to each eigenvalue is

$$\boldsymbol{v}_1 = \begin{bmatrix}3\\ 1\end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix}1\\ -3\end{bmatrix}. \tag{34}$$

The matrix of eigenvectors that defines the similarity transformation (11) is

$$\boldsymbol{P} = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2\end{bmatrix} = \begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix}. \tag{35}$$

The inverse of $\boldsymbol{P}$ is

$$\boldsymbol{P}^{-1} = \frac{1}{10} = \begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix}. \tag{36}$$

This yields the analytical solution

$$\begin{bmatrix}X_1(t,\boldsymbol{x}_0)\\ X_2(t,\boldsymbol{x}_0)\end{bmatrix} = \underbrace{\begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix}e^{3t} & 0\\ 0 & e^{-7t}\end{bmatrix}}_{e^{t\boldsymbol{\Lambda}}} \underbrace{\frac{1}{10}\begin{bmatrix}3 & 1\\ 1 & -3\end{bmatrix}}_{\boldsymbol{P}^{-1}} \underbrace{\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}}_{\boldsymbol{x}_0} \tag{37}$$

Developing the matrix products yields the desired flow

$$\begin{cases} X_1(t,\boldsymbol{x}_0) = \dfrac{x_{01}}{10}\left(9e^{3t} + e^{-7t}\right) + \dfrac{x_{02}}{10}\left(e^{3t} + 9e^{-7t}\right)\\[2ex] X_2(t,\boldsymbol{x}_0) = \dfrac{x_{01}}{10}\left(3e^{3t} - 3e^{-7t}\right) + \dfrac{x_{02}}{10}\left(3e^{3t} - 3e^{-7t}\right) \end{cases} \tag{38}$$

The phase portrait of this flow is shown in Figure 1.

Stable spiral. Consider the linear dynamical system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{39}$$

The eigenvalues of the matrix $\boldsymbol{A}$ are

$$\lambda_1 = -1 + i, \qquad \lambda_2 = -1 - i. \tag{40}$$

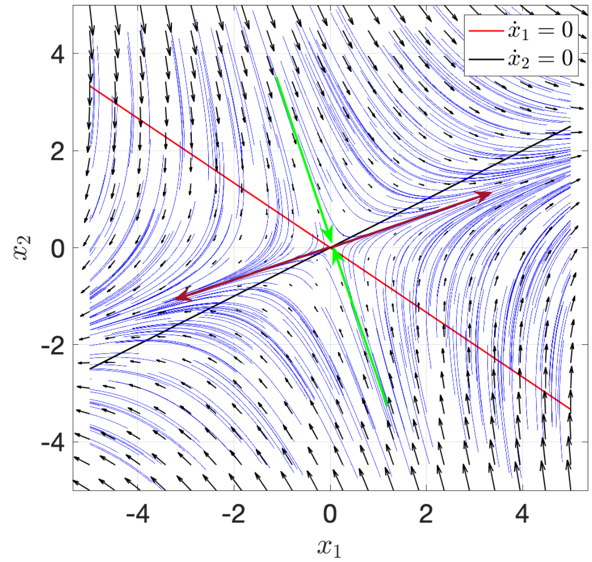

Figure 1: Saddle node. Shown are the nullclines, and the unstable (red arrows)/stable (green arrows) manifolds of the saddle identified by the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively.

These eigenvalues are complex conjugates and both have algebraic multiplicity one (simple eigenvalues), which implies that they have geometric multiplicity one. Therefore the matrix $\boldsymbol{A}$ is diagonalizable, and there exits a one-dimensional eigenspace (spanned by a complex vector) for each λ_i. To compute such eigenspaces/eigenvectors we proceed as usual

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} -iv_{11} = v_{12} \\ v_{12} \text{ or } v_{11} \text{ free} \end{cases} \tag{41}$$

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} iv_{21} = v_{22} \\ v_{21} \text{ or } v_{22} \text{ free} \end{cases} \tag{42}$$

We choose $v_1 = v_{21} = i$, which yields the following basis for the (complex) eigenspaces corresponding to λ_1 and λ_2, respectively

$$\boldsymbol{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}. \tag{43}$$

The similarity matrix $\boldsymbol{P}$ and its inverse are

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}, \qquad \boldsymbol{P}^{-1} = \frac{1}{2} \begin{bmatrix} -i & 1 \\ -i & 1 \end{bmatrix}. \tag{44}$$

The matrix exponential (17) is easily obtained as

$$\begin{aligned} e^{t\boldsymbol{A}} &= \underbrace{\begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} e^{t(-1+i)} & 0 \\ 0 & e^{t(-1-i)} \end{bmatrix}}_{e^{t\boldsymbol{\Lambda}}} \underbrace{\frac{1}{2} \begin{bmatrix} -i & 1 \\ -i & 1 \end{bmatrix}}_{\boldsymbol{P}^{-1}} \\ &= \frac{e^{-t}}{2} \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -ie^{it} & e^{it} \\ -ie^{-it} & -e^{-it} \end{bmatrix} \\ &= \frac{e^{-t}}{2} \begin{bmatrix} e^{it} + e^{-it} & ie^{it} - ie^{-it} \\ -ie^{it} + ie^{-it} & e^{it} + e^{-it} \end{bmatrix}. \end{aligned} \tag{45}$$

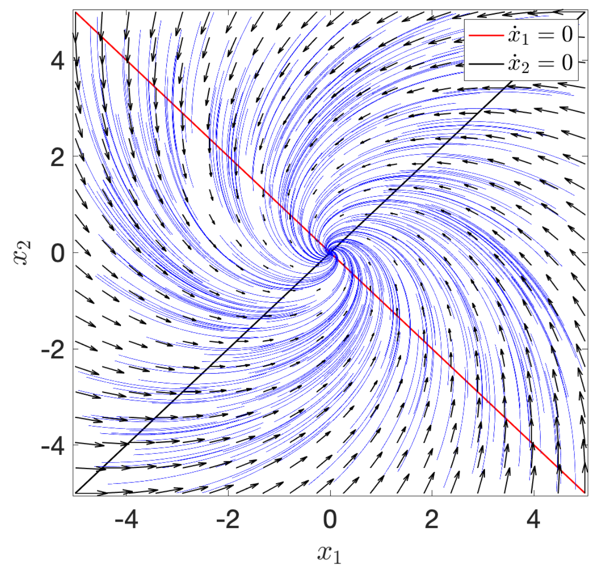

Figure 2: Stable spiral.

At this point we use the Euler formulas

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}, \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}, \tag{46}$$

to obtain

$$e^{t\boldsymbol{A}} = e^{-t} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}. \tag{47}$$

Applying $e^{t\boldsymbol{A}}$ to the initial condition $\boldsymbol{x}_0$ gives us the analytical solution

$$\begin{cases} X_1(t, \boldsymbol{x}_0) = e^{-t} \left[\cos(t) x_{01} - \sin(t) x_{02}\right] \\ X_2(t, \boldsymbol{x}_0) = e^{-t} \left[\sin(t) x_{01} + \cos(t) x_{02}\right] \end{cases}. \tag{48}$$

The phase portrait of this flow is shown in Figure 2.

- **Center.** Consider the linear dynamical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{49}$$

The eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = i, \qquad \lambda_2 = -i. \tag{50}$$

The eigenspaces associated with λ_1 and λ_2 are both one-dimensional (both eigenvalues are simple). Let us compute a basis for the eigenspace associated with λ_1

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} i v_{11} = v_{12} \\ v_{11} \text{ or } v_{12} \text{ free} \end{cases}. \tag{51}$$

We choose $v_{11} = 1$, which yields

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}. \tag{52}$$

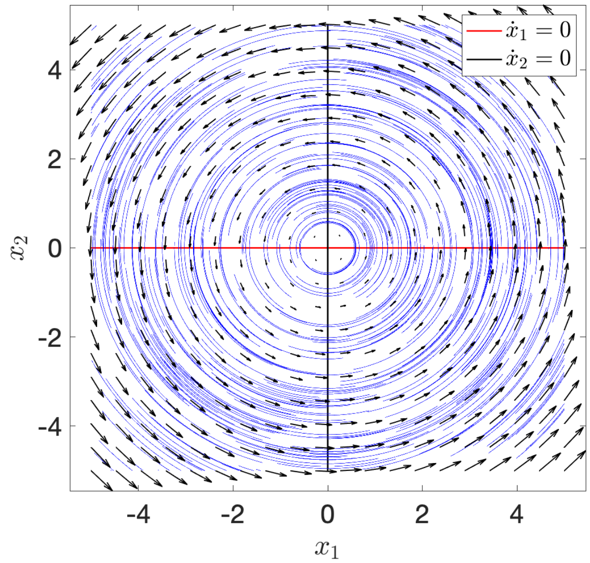

Figure 3: Center.

Similarly, for the eigenspace associated with λ_2 we have

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} iv_{21} = -v_{22} \\ v_{21} \text{ or } v_{22} \text{ free} \end{cases}. \tag{53}$$

We choose $v_{21} = 1$, which yields

$$\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{54}$$

The similarity matrix $\boldsymbol{P}$ and its inverse are

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \qquad \boldsymbol{P}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}. \tag{55}$$

The matrix exponential $e^{t\boldsymbol{A}}$ can be computed using equation (17)

$$\begin{aligned} e^{t\boldsymbol{A}} &= \underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}}_{e^{t\boldsymbol{\Lambda}}} \underbrace{\frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}}_{\boldsymbol{P}^{-1}} \\ &= \frac{1}{2} \begin{bmatrix} e^{it} + e^{-it} & -i\left(e^{it} - e^{-it}\right) \\ i\left(e^{it} - e^{-it}\right) & e^{it} + e^{-it} \end{bmatrix} \qquad (56) \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}, \qquad (57) \end{aligned}$$

where we used again the Euler formulas (46). A substitution of the matrix exponential into (10) yields the analytical solution

$$\begin{cases} X_1(t, \boldsymbol{x}_0) = x_{01} \cos(t) + x_{02} \sin(t) \\ X_2(t, \boldsymbol{x}_0) = -x_{01} \sin(t) + x_{02} \cos(t) \end{cases}. \tag{58}$$

The phase portrait of this flow is shown in Figure 3.

- **Degenerate node.** Consider the linear dynamical system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}}_{\boldsymbol{A}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{59}$$

The matrix $\boldsymbol{A}$ has only one eigenvalue $\lambda = 2$ with algebraic multiplicity 2. The dimension of the corresponding eigenspace, i.e., the dimension of the nullspace of $(\boldsymbol{A} - \lambda\boldsymbol{I})$ (geometric multiplicity of λ), can be calculated using the matrix rank theorem

$$\dim\left(N(\boldsymbol{A} - \lambda\boldsymbol{I})\right) = 2 - \text{rank}(\boldsymbol{A} - \lambda\boldsymbol{I}) = 2 - \underbrace{\text{rank}\left(\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\right)}_{=1} = 1 \tag{60}$$

Hence the dimension of the eigenspace associated with $\lambda = 2$, is equal to one. This implies that the matrix $\boldsymbol{A}$ is *not* diagonalizable. Let us compute a basis for the one-dimensional eigenspace. We have

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \Leftrightarrow \quad \begin{cases}v_{11} = v_{12}\\ v_{11} \text{ or } v_{12} \text{ free}\end{cases} \tag{61}$$

We choose $v_{12} = 1$, which yields

$$\boldsymbol{v}_1 = \begin{bmatrix}1\\ 1\end{bmatrix}. \tag{62}$$

At this point we need to complement $\boldsymbol{v}_1$ to a basis of $\mathbb{R}^2$ by adding one linearly independent vector. To this end, we compute the so-called generalized eigenvector[4] by solving the linear equation

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1 \tag{64}$$

We obtain

$$\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix} \qquad \Rightarrow \qquad \begin{cases}-v_{21} + v_{22} = 1\\ v_{21} \text{ or } v_{22} \text{ free}\end{cases} \tag{65}$$

We choose $v_{22} = 1$ which gives the generalized eigenvector

$$\boldsymbol{v}_2 = \begin{bmatrix}0\\ 1\end{bmatrix}. \tag{66}$$

The similarity matrix in this case has the eigenvector $\boldsymbol{v}_1$ and the generalized eigenvector $\boldsymbol{v}_2$ as columns

$$\boldsymbol{P} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \qquad \Leftrightarrow \qquad \boldsymbol{P}^{-1} = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}. \tag{67}$$

The matrix exponential of the Jordan block that corresponds to the eigenvalue $\lambda = 2$ with algebraic multiplicity two and geometric multiplicity one is (see Table 1)

$$e^{t\boldsymbol{J}} = \begin{bmatrix}e^{2t} & te^{2t}\\ 0 & e^{2t}\end{bmatrix}. \tag{68}$$

[4]Note that the generalized eigenvector $\boldsymbol{v}_2$ defined in (64) is in the nullspace of the matrix $(\boldsymbol{A} - \lambda\boldsymbol{I})^2$. In fact,

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{v}_1 \quad \Rightarrow (\boldsymbol{A} - \lambda\boldsymbol{I})^2\boldsymbol{v}_2 = \boldsymbol{0}_{\mathbb{R}^2}. \tag{63}$$

It can be shown that eigenvectors and generalized eigenvectors are linearly independent.

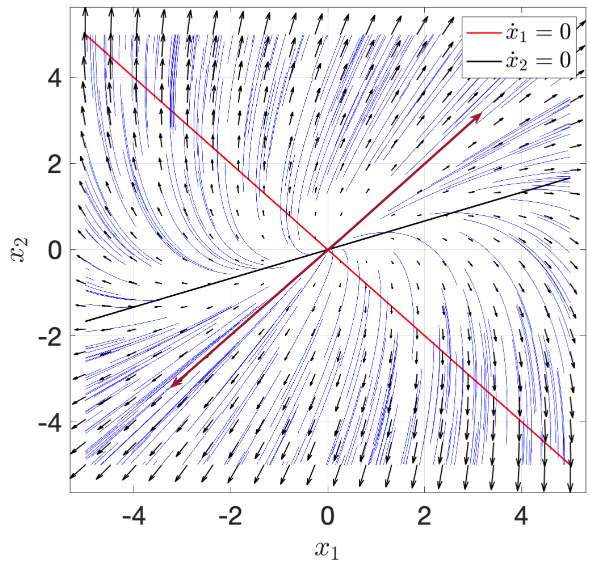

Figure 4: Degenerate node. Shown is the unstable manifold of the node (red arrows), which is defined by the eigendirection $\boldsymbol{v}_1$ corresponding to the eigenvalue $\lambda = 2$.

The the matrix exponential $e^{t\boldsymbol{A}}$ can now be computed explicitly via the formula (24)

$$
\begin{aligned}
e^{t\boldsymbol{A}} &= \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}}_{e^{t\boldsymbol{J}}} \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{\boldsymbol{P}^{-1}} \\
&= \begin{bmatrix} e^{2t} - te^{2t} & te^{2t} \\ -te^{2t} & e^{2t} + te^{2t} \end{bmatrix}.
\end{aligned} \tag{69}
$$

This gives the analytical solution

$$
\begin{cases} X_1(t, \boldsymbol{x}_0) = \left(e^{2t} - te^{2t}\right) x_{01} + te^{2t} x_{02} \\ X_2(t, \boldsymbol{x}_0) = -te^{2t} x_{01} + \left(e^{2t} + te^{2t}\right) x_{02} \end{cases}. \tag{70}
$$

The phase portrait of this flow is shown in Figure 4.

Classification of two-dimensional flows generated linear dynamical systems. In Figure 5 and Figure 6 we provide a classification of all possible flows generated by two-dimensional dynamical systems in terms of the eigenvalues of the matrix $\boldsymbol{A}$. Of course, changing the sign of the eigenvalues of $\boldsymbol{A}$ is equivalent to transforming the matrix from $\boldsymbol{A}$ to $-\boldsymbol{A}$. This yields an inversion in the orientation of all trajectories, which implies, e.g., that stable nodes become unstable, centers spin the other way around, etc.

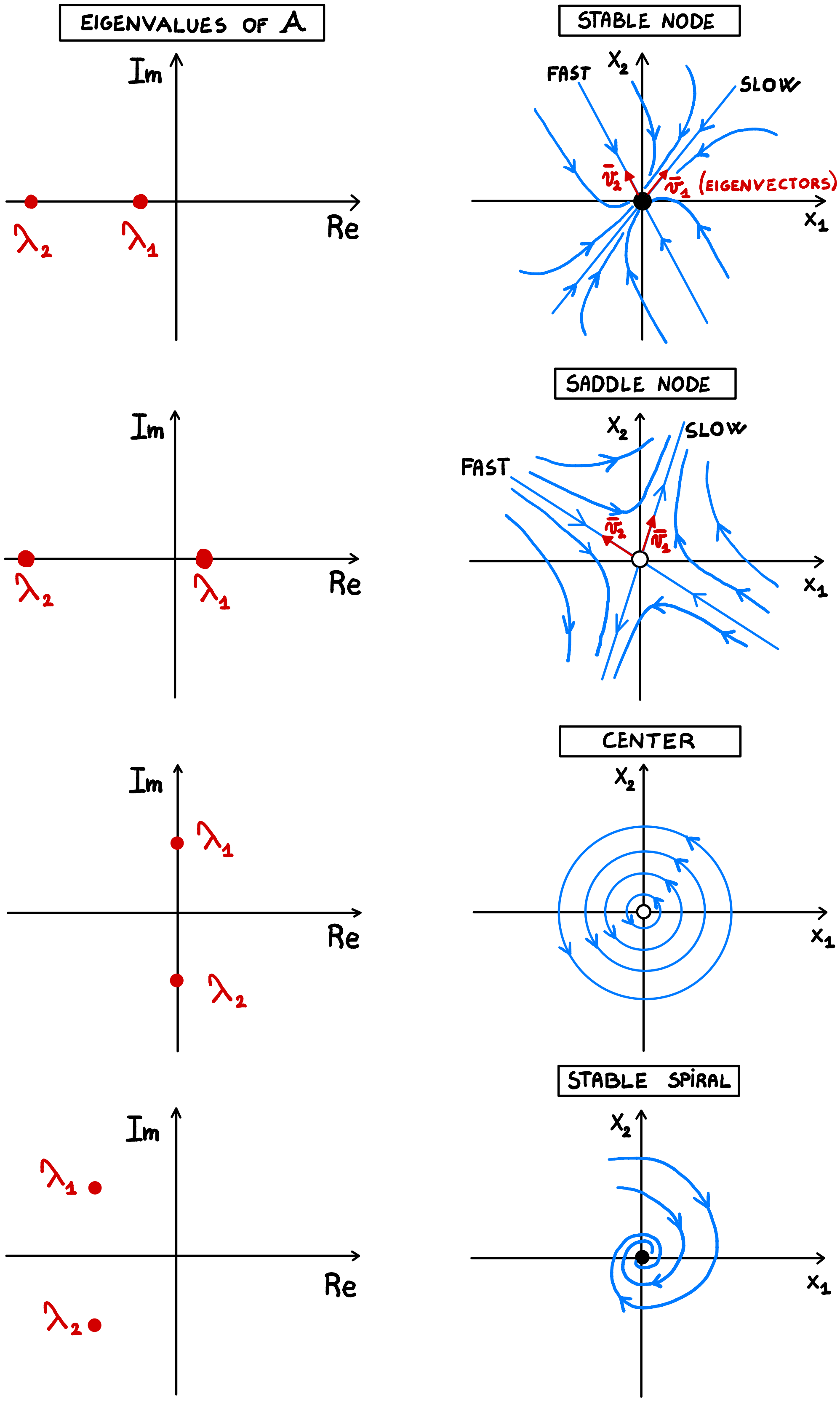

Figure 5: Classification of flows generated by two-dimensional dynamical systems in terms of the eigenvalues of the matrix $\boldsymbol{A}$.

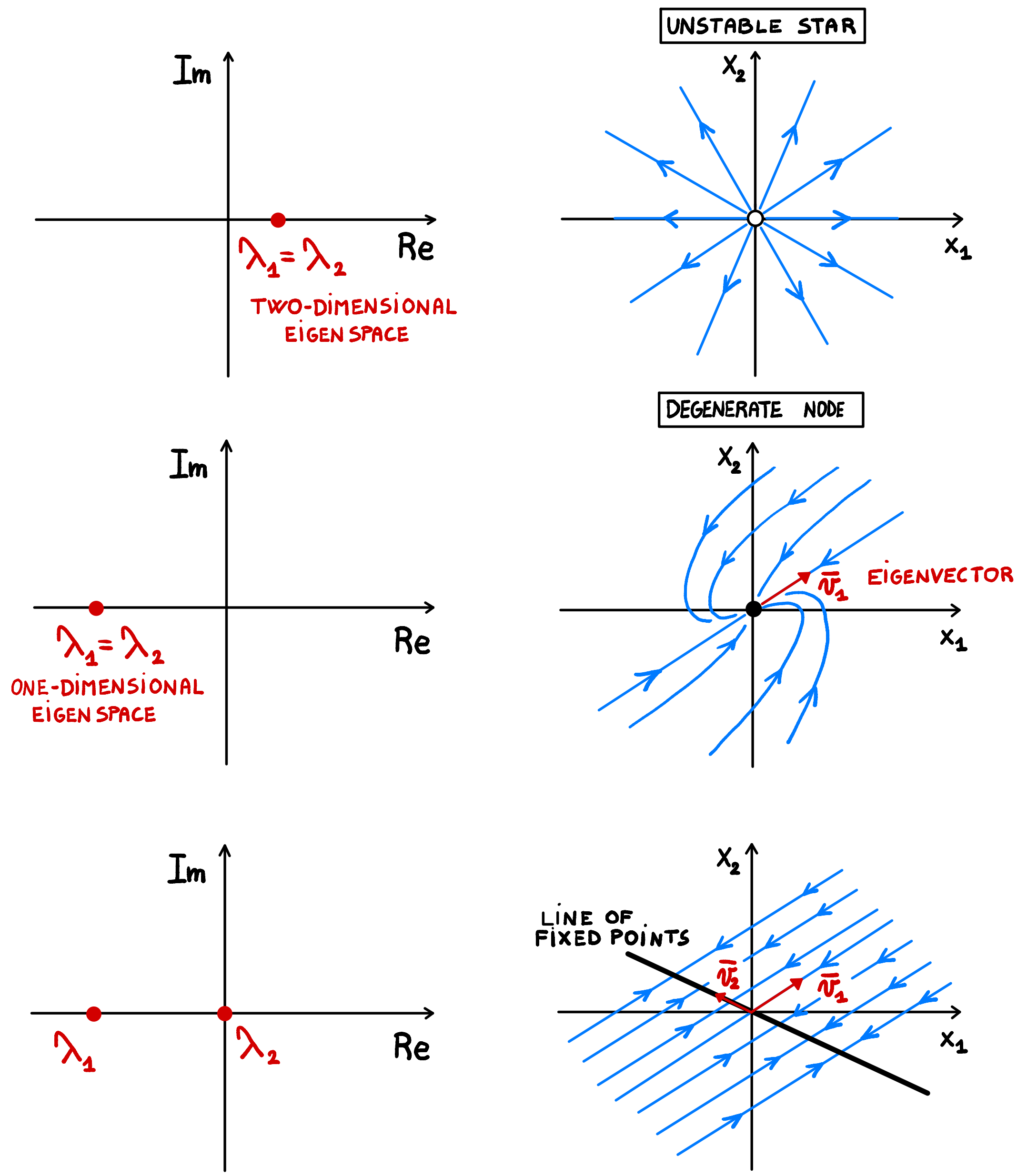

Figure 6: Classification of flows generated by two-dimensional dynamical systems in terms of the eigenvalues of the matrix $\boldsymbol{A}$.

Three-dimensional linear dynamical systems

In this section, we derive the analytical expression for the flow generated by linear systems in three dimensions. The methods presented here extend naturally to higher-dimensional systems.

Example: Consider the three dimensional linear system Consider the linear dynamical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{71}$$

The matrix $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ (with algebraic multiplicity two) and $\lambda_2 = -1$ (with algebraic multiplicity one). The dimension of the eigenspace corresponding to λ_1, i.e., the geometric multiplicity of λ_1 is

$$\dim(N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})) = 3 - \text{rank}(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 3 - \text{rank}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}\right) = 3 - 1 = 2. \tag{72}$$

Therefore the matrix $\boldsymbol{A}$ is diagonalizable. The eigenvectors corresponding to λ_1 are solution to the linear system $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^3}$, i.e.,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} v_1 + v_2 - 2v_3 = 0 \\ (v_1, v_2) \text{ or } (v_1, v_3) \text{ or } (v_2, v_3) \text{ are arbitrary} \end{cases} \tag{73}$$

We pick $(v_2, v_3) = (1, 1)$ and $(v_2, v_3) = (2, 1)$ which yields the following eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}. \tag{74}$$

Any linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is still an eigenvector. The eigenvectors corresponding to $\lambda_2 = -1$ are solutions to the linear system $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^3}$, i.e.,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} v_1 = 0 \\ v_2 = 0 \\ v_3 \text{ is arbitrary} \end{cases} \tag{75}$$

We choose

$$\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{76}$$

The similarity matrix and its inverse are

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{P}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \tag{77}$$

Therefore, the analytical solution of the 3D linear system (71) is

$$\begin{bmatrix} X_1(t, \boldsymbol{x}_0) \\ X_2(t, \boldsymbol{x}_0) \\ X_3(t, \boldsymbol{x}_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{01} \\ x_{03} \end{bmatrix} \tag{78}$$

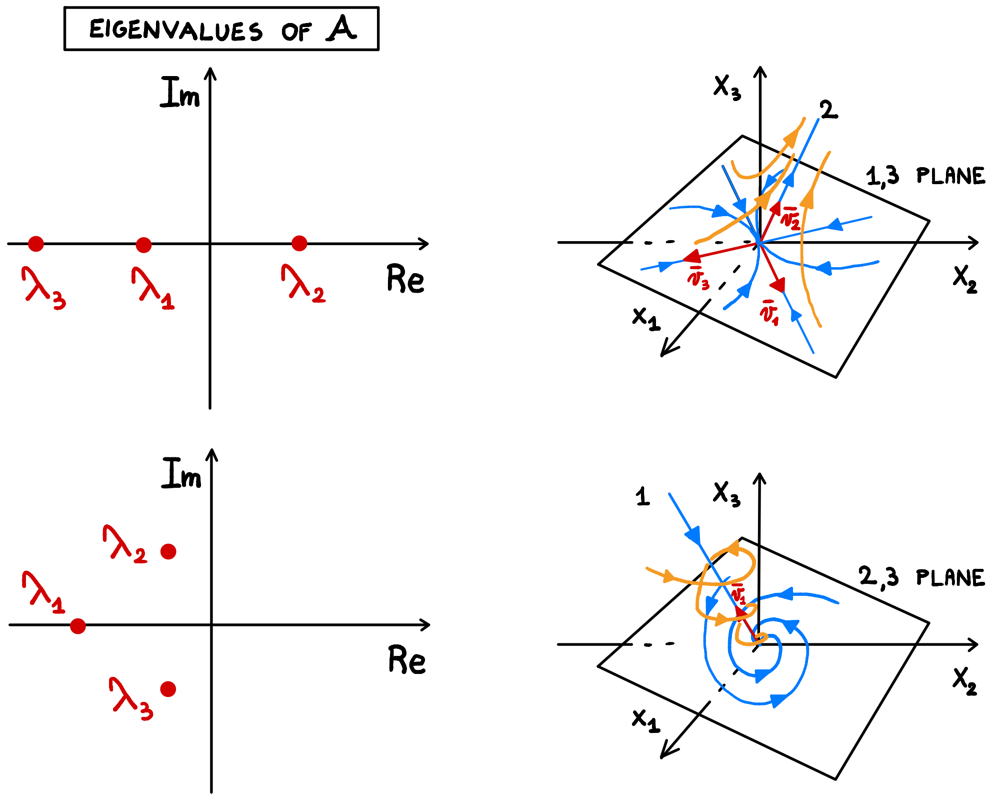

Figure 7: Examples of flows generated by three-dimensional dynamical systems in terms of the eigenvalues of the matrix $\boldsymbol{A}$. The 2-3 plane is defined by two real three-dimensional vectors obtained from the *real* Jordan form of the matrix $\boldsymbol{A}$ (see Appendix B).

i,e,

$$\begin{cases} X_1(t, \boldsymbol{x}_0) = e^t x_{01} \\ X_2(t, \boldsymbol{x}_0) = e^t x_{02} \\ X_3(t, \boldsymbol{x}_0) = \dfrac{(e^t - e^{-t})}{2}(x_{01} + x_{02}) + e^{-t} x_{03} \end{cases} \tag{79}$$

Three- and higher-dimensional flows generated linear dynamical systems In Figure 7 we provide a few sketches of three-dimensional flows corresponding to matrices $\boldsymbol{A}$ with various eigenvalues. As easily seen, the classification of these flows is not as straightforward as in the 2D case. In fact, we can have spiraling directions, saddle node planes, etc.

Example: Consider the following initial-boundary value problem for the heat equation in the periodic spatial domain $[0, 2\pi]$

$$\begin{cases} \dfrac{\partial u(t,y)}{\partial t} = \alpha \dfrac{\partial^2 u(y,t)}{\partial y^2} & \text{heat equation} \\ u(0,y) = u_0(y) & \text{initial condition} \\ u(t,0) = u(t,2\pi) & \text{periodic boundary conditions} \end{cases} \tag{80}$$

This equation describes heat propagation by thermal conduction in a one-dimensional slab with periodic conditions. We have seen that a finite difference approximation of this problem on the evenly-spaced grid with n spatial points

$$y_k = (k-1)\Delta y \qquad k = 1, \ldots, n, \qquad \Delta y = \frac{2\pi}{n} \tag{81}$$

yields the n-dimensional *linear dynamical system*

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}, \tag{82}$$

where $x_j(t) \simeq u(t, y_j)$ and

$$\boldsymbol{A} = \frac{\alpha}{\Delta y^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 1 & \cdots & 0 & 1 & -2 \end{bmatrix} \tag{83}$$

It can be shown that the eigenvalues of the matrix $\boldsymbol{A}$ (circulant Laplacian) are

$$\lambda_j = -\frac{4\alpha}{\Delta y^2} \sin^2\left(\frac{\pi j}{n}\right) \qquad j = 0, \ldots, n-1. \tag{84}$$

Since $\lambda_0 = 0$, we have that $\boldsymbol{A}$ is not invertible. Indeed, there exists a one-dimensional nullspace spanned by the unit vector

$$\boldsymbol{v}_0 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T. \tag{85}$$

This implies that this system has an infinite number of fixed points $\boldsymbol{x}^* = \kappa \boldsymbol{v}_0$. Physically, this is saying that there is no temperature dynamics in a homogeneous one-dimensional slab with periodic boundary conditions and isothermal initial condition.

Appendix A: The matrix eigenvalue problem

In this Appendix we briefly review the eigenvalue problem for a $n \times n$ matrix $\boldsymbol{A}$ with real coefficients. The eigenvalue problem is essentially the problem of finding all real (or complex) numbers λ (eigenvalues) and all nonzero real (or complex) vectors $\boldsymbol{v}$ (eigenvectors) satisfying the equation

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}. \tag{86}$$

Computation of eigenvalues. From equation (86) it follows that

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^n}, \tag{87}$$

Hence, the eigenvector v (which is non-zero by definition) is in the nullspace of the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$. This implies that the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$ is not invertible[5]. A necessary and sufficient condition for $(\boldsymbol{A} - \lambda \boldsymbol{I})$ to be not invertible is

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \qquad \text{(characterististic equation)}. \tag{88}$$

The polynomial

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) \tag{89}$$

is known as *characteristic polynomial* associated with the matrix $\boldsymbol{A}$. The characteristic equation (88) implies that the eigenvalues of the matrix $\boldsymbol{A}$ are roots of the characteristic polynomial $p(\lambda)$.

How many eigenvalues do we have for a given $n\times n$ matrix $\boldsymbol{A}$? The characteristic polynomial $p(\lambda)$ associated with the matrix $\boldsymbol{A}$ is a polynomial of degree n with real coefficients. Hence, by using the fundamental theorem of algebra we conclude $p(\lambda)$ has exactly n roots which may be real or complex conjugates. In other words, every $n \times n$ matrix has exactly n eigenvalues. Such eigenvalues may be repeated, in which case we say that they have "algebraic multiplicity" greater than one. In other words, the multiplicity of an eigenvalue as a root of the characteristic polynomial is called *algebraic multiplicity* the eigenvalue.

Example 1: Compute the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}. \tag{90}$$

The characteristic polynomial is

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = -(2-\lambda)(6+\lambda) - 9, \tag{91}$$

i.e.,

$$p(\lambda) = \lambda^2 + 4\lambda - 21. \tag{92}$$

The eigenvalues of $\boldsymbol{A}$ are roots of $p(\lambda)$. Setting $p(\lambda) = 0$ yields

$$\lambda_{1,2} = -2 \pm \sqrt{4+21} = -2 \pm 5 \qquad \Rightarrow \qquad \lambda_1 = 3, \quad \lambda_2 = -7. \tag{93}$$

In this case, both eigenvalues have algebraic multiplicity one, i.e., they are simple roots of $p(\lambda)$. The characteristic polynomial can be factored as

$$p(\lambda) = (\lambda - 3)(\lambda + 7), \tag{94}$$

suggesting once again that $\lambda = 3$ and $\lambda = -7$ are simple roots.

[5] The matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$ in (87) maps a non-zero vector $\boldsymbol{v}$ into $\boldsymbol{0}_{\mathbb{R}^n}$. Hence the the nullspace of $(\boldsymbol{A} - \lambda \boldsymbol{I})$ has a nonzero vector in it, which implies that the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$ is not invertible.

Example 2: Compute the eigenvalues of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 & 1 & -5 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{95}$$

In this case we have

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 2-\lambda & 5 & 1 & -5 \\ 0 & 4-\lambda & 3 & 0 \\ 0 & 0 & 2-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \tag{96}$$

and

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = (2-\lambda)^2(4-\lambda)(1-\lambda). \tag{97}$$

Hence, the matrix $\boldsymbol{A}$ has three eigenvalues:

$$\begin{aligned} \lambda_1 &= 2 \qquad \text{with algebraic multiplicity 2,} \\ \lambda_2 &= 4 \qquad \text{with algebraic multiplicity 1,} \\ \lambda_3 &= 1 \qquad \text{with algebraic multiplicity 1.} \end{aligned}$$

Note that the eigenvalues coincides with the diagonal entries of the matrix $\boldsymbol{A}$. This is a general fact about upper or or lower triangular matrices, i.e., the eigenvalues of such matrices coincides with the diagonal entries of the matrix. For example, the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{98}$$

has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$, both with algebraic multiplicity 2.

Example 3: Compute the eigenvalues of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}. \tag{99}$$

The characteristic polynomial is

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 1-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix} = -(1-\lambda)^2 + 2, \tag{100}$$

i.e.,

$$p(\lambda) = \lambda^2 - 2\lambda + 3. \tag{101}$$

Hence, the eigenvalues are

$$\lambda_1 = 1 + i\sqrt{2} \qquad \lambda_2 = 1 - i\sqrt{2} \tag{102}$$

Note that λ_1 and λ_2 are complex conjugates eigenvalues. Clearly, for 2×2 matrices with real entries the fundamental theorem of algebra tells us that the eigenvalues are either both real or complex conjugates.

Eigenvectors and eigenspaces. By definition, an eigenvector of a $n \times n$ matrix $\boldsymbol{A}$ is a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}. \tag{103}$$

This means that $\boldsymbol{v}$ is a nonzero vector in the nullspace of the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$. In fact, $\boldsymbol{v}$ is mapped onto the zero of $\mathbb{R}^n$ by $(\boldsymbol{A} - \lambda \boldsymbol{I})$ (see equation (87)). We know that the nullspace of a $n \times n$ matrix is a vector subspace of $\mathbb{R}^n$.

We denote by $N(\boldsymbol{A} - \lambda \boldsymbol{I})$ the nullspace of the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})$, and call $N(\boldsymbol{A} - \lambda \boldsymbol{I})$ the *eigenspace* of $\boldsymbol{A}$ corresponding to the eigenvalue λ. The dimension of the eigenspace $N(\boldsymbol{A} - \lambda \boldsymbol{I})$ is called *geometric multiplicity* of the eigenvalue λ. By definition, an eigenvector cannot be zero and therefore the eigenspace corresponding to each eigenvalue has dimension at least equal to one. The dimension of the eigenspace corresponding to some eigenvalue can be computed by using the matrix rank theorem.

Example 4: Compute the eigenspaces of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \tag{104}$$

We have seen in Example 1 that the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 3$ and $\lambda_2 = -7$. Let us compute the eigenspace corresponding to λ_1. To this end, we first compute the dimension of such eigenspace by using the matrix rank theorem

$$\dim(N(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 2 - \operatorname{rank}(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 2 - \operatorname{rank}\left(\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}\right) = 2 - 1 = 1 \tag{105}$$

Hence, the eigenspace corresponding to λ_1 has dimension one. Any vector of such an eigenspace is an eigenvector of A corresponding to λ_1. To compute a basis for the eigenspace $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$ consider

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad -v_1 + 3v_2 = 0 \tag{106}$$

Hence,

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{107}$$

is a basis for $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$, and an eigenvector of $\boldsymbol{A}$ corresponding to λ_1. All eigenvectors of $\boldsymbol{A}$ corresponding to λ_1 are in the form

$$c \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{with } c \neq 0. \tag{108}$$

Similarly, the eigenspace corresponding to λ_2 has dimension 1 and can be determined by solving the linear system

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad 3v_1 + v_2 = 0. \tag{109}$$

Hence,

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \tag{110}$$

is a basis for $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$ and an eigenvector of $\boldsymbol{A}$ corresponding to λ_2. In summary, λ_1 and λ_2 are eigenvalues with algebraic multiplicity one and geometric multiplicity one. Geometric multiplicity one means that the eigenspaces $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$ and $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$ are both one-dimensional. A basis for $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$ and $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$ is given by (107) and (110), respectively.

The following theorem establishes a relationship between the algebraic multiplicity and the geometric multiplicity of an eigenvalue λ.

Theorem 1. Let λ be an eigenvalue of a $n \times n$ matrix $\boldsymbol{A}$. Denote by s the algebraic multiplicity of λ. Then

$$\dim(N(\boldsymbol{A} - \lambda \boldsymbol{I})) \leq s. \tag{111}$$

In other words the geometric multiplicity of the eigenvalue λ (i.e., the dimension of the associated eigenspace) is always smaller or equal than the algebraic multiplicity).

Of course, if λ is a simple eigenvalue ($s = 1$) then $\dim(N(\boldsymbol{A} - \lambda \boldsymbol{I})) = 1$, i.e., the eigenspace corresponding to simple eigenvalues is always one-dimensional. If λ has algebraic multiplicity 2, i.e., it is a repeated eigenvalue, then it is possible to have geometric multiplicity equal to one or equal to two. In the latter case the eigenspace is two-dimensional and any vector in such eigenspace (including linear combinations of multiple eigenvectors) is an eigenvector. Let us provide a simple example of a 2×2 matrix with one eigenvalue of algebraic multiplicity two and geometric multiplicity one.

Example 5: Consider the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}. \tag{112}$$

We know that $\lambda = 2$ is the only eigenvalue and it has algebraic multiplicity two. In fact, the characteristic polynomial is $p(\lambda) = (2 - \lambda)^2$. The geometric multiplicity of $\lambda = 2$ can be calculated by using the matrix rank theorem

$$\dim(N(\boldsymbol{A} - \lambda \boldsymbol{I})) = 2 - \operatorname{rank}(\boldsymbol{A} - \lambda \boldsymbol{I}) = 2 - \underbrace{\operatorname{rank}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)}_{=1} = 2 - 1 = 1. \tag{113}$$

Hence, the eigenspace associated with $\lambda = 2$ is one-dimensional. A basis for such an eigenspace is obtained as follows:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad v_2 = 0. \tag{114}$$

We choose

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{115}$$

Example 6: Compute the eigenvalues and the eigenvectors of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}. \tag{116}$$

This is an upper triangular matrix and therefore the eigenvalues coincide with the diagonal entries. Hence we have $\lambda_1 = 2$ with algebraic multiplicity two and $\lambda_2 = 1$ with algebraic multiplicity one.

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \Leftrightarrow \quad \dim(N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})) = 3 - \underbrace{\operatorname{rank}(\boldsymbol{A} - \lambda_1 \boldsymbol{I})}_{=2} = 1, \tag{117}$$

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \quad \Leftrightarrow \quad \dim(N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})) = 3 - \underbrace{\operatorname{rank}(\boldsymbol{A} - \lambda_2 \boldsymbol{I})}_{=2} = 1. \tag{118}$$

Therefore, the dimension of the eigenspaces associated with λ_1 and λ_2 is one. Let us find a basis for such eigenspaces.

$$(\boldsymbol{A}-\lambda_1\boldsymbol{I})\boldsymbol{v}=\boldsymbol{0}_{\mathbb{R}^3} \quad \Rightarrow \quad \begin{bmatrix}0 & 1 & 3\\ 0 & -1 & 5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_1 \text{ arbitrary}\\ v_2+3v_3=0\\ -v_2+5v_3=0\end{cases} \tag{119}$$

Hence, an eigenvector that spans $N(\boldsymbol{A}-\lambda_1\boldsymbol{I})$ is

$$\boldsymbol{v}=\begin{bmatrix}1\\0\\0\end{bmatrix}. \tag{120}$$

Similarly,

$$(\boldsymbol{A}-\lambda_2\boldsymbol{I})\boldsymbol{v}=0_{\mathbb{R}^3} \quad \Rightarrow \quad \begin{bmatrix}1 & 1 & 3\\ 0 & 0 & 5\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_1+v_2+3v_3=0\\ v_3=0\end{cases} \tag{121}$$

Hence, an eigenvector that spans $N(\boldsymbol{A}-\lambda_2\boldsymbol{I})$ is

$$\boldsymbol{v}=\begin{bmatrix}1\\-1\\0\end{bmatrix}. \tag{122}$$

Hereafter, we recall an important theorem on eigenvectors corresponding to different eigenvalues.

Theorem 2. Eigenvectors corresponding to different eigenvalues are linearly independent.

Of course if an eigenvalue λ has geometric multiplicity larger than one, then we can construct a basis for $N(\boldsymbol{A}-\lambda\boldsymbol{I})$. In any case, such basis will be linearly independent on any other eigenvector corresponding to a different eigenvalue.

Similarity transformations. Let $\boldsymbol{A},\boldsymbol{B}\in M_{n\times n}(\mathbb{R}^n)$. We say that $\boldsymbol{A}$ is *similar* to $\boldsymbol{B}$ is there exists an invertible matrix $\boldsymbol{P}\in M_{n\times n}(\mathbb{R}^n)$ such that

$$\boldsymbol{A}\boldsymbol{P}=\boldsymbol{P}\boldsymbol{B} \qquad \Leftrightarrow \qquad \boldsymbol{A}=\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1} \tag{123}$$

The transformation $\boldsymbol{B}\to\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$ is called *similarity transformation.* An example of similarity transformation is the change of basis transformation.

Theorem 3. Similar matrices have the same eigenvalues.

Proof. Let $A,B\in M_{n\times n}$ be two similar matrices, i.e., $P\in M_{n\times n}$ such that

$$A=PBP^{-1}. \tag{124}$$

Then

$$\det(A-\lambda I)=\det(PBP^{-1}-\lambda PP^{-1})=\det(P)\det(B-\lambda I)\det(P^{-1})=\det(B-\lambda I) \tag{125}$$

□

Diagonalization. Consider a $n \times n$ matrix $\boldsymbol{A}$. We have seen in Theorem 2 that eigenvectors corresponding to different eigenvalues are linearly independent. Hence, if the algebraic multiplicity of each eigenvalue is equal to the geometric multiplicity then it is possible to construct a basis for $\mathbb{R}^n$ made of eigenvectors of A. Let us organize such n eigenvectors as columns of a matrix P

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{bmatrix}. \tag{126}$$

Clearly,

$$\boldsymbol{AP} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{v}_1 & \cdots & \boldsymbol{A}\boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}}_{\Lambda} = \boldsymbol{P\Lambda}, \tag{127}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with the eigenvalues of A (counted with their multiplicity) sitting along the diagonal. Equation (127) shows that if $\boldsymbol{A}$ has n linearly independent eigenvectors then $\boldsymbol{A}$ is similar to a diagonal matrix[6] $\boldsymbol{\Lambda}$. The similarity transformation is defined by the matrix $\boldsymbol{P}$ in (126), i.e., the matrix that has the eigenvectors of $\boldsymbol{A}$ as columns.

A simple corollary of this statement is that matrices with simple eigenvalues are always diagonalizable, since they have n linearly independent eigenvectors.

Theorem 4. Let $\boldsymbol{A}$ be a $n \times n$ matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$ with algebraic multiplicities $\{s_1, \ldots, s_p\}$, respectively. Then $\boldsymbol{A}$ is diagonalizable if and only if

$$\dim(N(\boldsymbol{A} - \lambda_i \boldsymbol{I})) = s_i \quad \text{for all} \quad i = 1, \ldots, p. \tag{128}$$

This theorem is saying that if each eigenvalue of a matrix $\boldsymbol{A}$ has algebraic multiplicity equal to its geometric multiplicity then the matrix $\boldsymbol{A}$ is similar to a diagonal matrix. Conversely, if a matrix $\boldsymbol{A}$ is similar to a diagonal matrix then each eigenvalue of $\boldsymbol{A}$ has algebraic multiplicity equal to its geometric multiplicity.

Example 7: The matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \tag{129}$$

is diagonalizable. In fact, we have seen that the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -7$ (simple eigenvalues). This implies that the dimension of the associated eigenspace is one for both eigenvalues. The eigenvectors of $\boldsymbol{A}$ are

$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \tag{130}$$

Define

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}, \qquad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}. \tag{131}$$

It is straightforward to verify that

$$\boldsymbol{P}^{-1} = \frac{1}{10} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \tag{132}$$

and

$$\boldsymbol{A} = \boldsymbol{P\Lambda P}^{-1} \qquad \text{or} \qquad \boldsymbol{\Lambda} = \boldsymbol{P}^{-1}\boldsymbol{AP}. \tag{133}$$

[6] In general, we say that a matrix $\boldsymbol{A}$ is *diagonalizable* if there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{A}$ is similar to a diagonal matrix.

Example 8: The matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \tag{134}$$

is *not* diagonalizable. In fact the algebraic multiplicity of the eigenvalue $\lambda = 2$ is two, while its geometric multiplicity is one. We will see hereafter that it is possible to complement the eigenvector that spans the eigenspace with another linearly independent vector called "generalized eigenvector" to form a basis of $\mathbb{R}^2$. Such generalized eigenvector of $\boldsymbol{A}$, makes $\boldsymbol{A}$ similar to a matrix $\boldsymbol{J}$ called *Jordan form* of $\boldsymbol{A}$. In this particular example, the Jordan form of $\boldsymbol{A}$ coincides with $\boldsymbol{A}$, i.e., $\boldsymbol{A}$ is already in a Jordan form.

Example 9: Verify that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \tag{135}$$

is diagonalizable. The matrix is lower-triangular with eigenvalues $\lambda_1 = 1$ (algebraic multiplicity two) and $\lambda_2 = 2$ (algebraic multiplicity one). To verify that A is diagonalizable we just need to check that the geometric multiplicity of $\lambda_1 = 1$ is equal to two. To this end, we use the matrix rank theorem:

$$\dim(N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})) = 3 - \operatorname{rank}(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 3 - \operatorname{rank}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = 3 - 1 = 2 \tag{136}$$

This shows that the dimension of the nullspace of $N(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$, i.e., the dimension of the eigenspace associated with $\lambda_1 = 1$ is two. Let us compute a basis for such an eigenspace. To this end,

$$(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^3} \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_1 \text{ arbitrary} \\ v_2 \text{ arbitrary} \\ v_3 = -v_2 \end{cases} \tag{137}$$

Hence, a basis for the eigenspace corresponding to λ_1 is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}. \tag{138}$$

On the other hand, the eigenspace $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$ is spanned by a vector that can be computed as

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^3} \quad \Rightarrow \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_1 = 0 \\ v_2 = 0 \\ v_3 \text{ arbitrary} \end{cases} \tag{139}$$

Therefore a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ is

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \tag{140}$$

Indeed, it can be verified by a direct calculation that

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\boldsymbol{\Lambda}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\boldsymbol{P}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\boldsymbol{P}}. \tag{141}$$

Appendix B: Generalized eigenvectors and Jordan canonical form

The set of eigenvectors of any $n \times n$ matrix $\boldsymbol{A}$ can be complemented to a basis of $\mathbb{R}^n$. To this end, we can add a certain number of so-called *generalized eigenvectors*, to each "defective" eigenspace of $\boldsymbol{A}$. A defective eigenspace of $\boldsymbol{A}$ is an eigenspace with dimension $\dim(N(\boldsymbol{A} - \lambda_i \boldsymbol{I}))$ smaller than the algebraic multiplicity s_i of the associated eigenvalue λ_i (see Theorem 1). For such defective eigenspaces we compute

$$s_i - \dim(N(\boldsymbol{A} - \lambda_i \boldsymbol{I})) \tag{142}$$

additional generalized eigenvectors. This yields a basis of $\mathbb{R}^n$ made of eigenvectors and generalized eigenvectors of $\boldsymbol{A}$. Such basis, also induces a similarity transformation between $\boldsymbol{A}$ and a matrix called *Jordan canonical form* of $\boldsymbol{A}$. Let us describe the procedure to compute the Jordan form of a matrix $\boldsymbol{A}$. To this end, let us first consider the simple 2 matrix

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{143}$$

We know that the eigenspace corresponding to the eigenvalue λ is one-dimensional with basis

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{144}$$

To complement $\boldsymbol{v}$ with another vector and form a basis of $\mathbb{R}^2$ we choose $\boldsymbol{w}$ as follows

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v}. \tag{145}$$

Clearly, $\boldsymbol{w}$ is in the nullspace of the matrix $(\boldsymbol{A} - \lambda \boldsymbol{I})^2$. In fact, by applying $(\boldsymbol{A} - \lambda \boldsymbol{I})$ to both sides of (145) we obtain

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 \boldsymbol{w} = (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^2}. \tag{146}$$

It can be shown that $\boldsymbol{w}$ and $\boldsymbol{v}$ are linearly independent. To compute the generalized eigenvector $\boldsymbol{w}$ we solve the linear system (145)

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{w} = \boldsymbol{v} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} w_1 \text{ arbitrary} \\ w_2 = 1 \end{cases} . \tag{147}$$

Hence a generalized eigenvector for the eigenspace $N(\boldsymbol{A} - \lambda \boldsymbol{I})$ is

$$\boldsymbol{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{148}$$

At this point we define the similarity transformation

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{149}$$

and apply $\boldsymbol{A}$ to $\boldsymbol{P}$ to obtain

$$\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{v} & \boldsymbol{A}\boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} \underbrace{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}_{\boldsymbol{J}} = \boldsymbol{P}\boldsymbol{J}. \tag{150}$$

Hence, $\boldsymbol{A}$ is similar to a matrix $\boldsymbol{J}$ in a particular form (not diagonal but almost diagonal), known as *Jordan canonical form* of $\boldsymbol{A}$. In this particular example, $\boldsymbol{A}$ is already in a Jordan form so the similarity transformation defined by $\boldsymbol{P}$ turns out to be the identity transformation.

Next, let us consider a 3×3 matrix $\boldsymbol{A}$ with only one eigenvalue λ of algebraic multiplicity three and geometric multiplicity two.

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 1 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{151}$$

The eigenspace of $\boldsymbol{A}$ corresponding to the eigenvalue λ is

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}_{\mathbb{R}^3} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_1 \text{ arbitrary} \\ v_2 \text{ arbitrary} \\ v_3 = -v_2 \end{cases}. \tag{152}$$

Hence a basis for $N(\boldsymbol{A} - \lambda \boldsymbol{I})$ is

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{153}$$

To complement $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to a basis of $\mathbb{R}^3$ we add a generalized eigenvector $\boldsymbol{v}_3$ that solves the following linear system[7]

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{v}_2. \tag{154}$$

We obtain

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} v_{31} \text{ arbitrary} \\ v_{32}, \text{ arbitrary} \\ v_{32} + v_{33} = 1 \end{cases}. \tag{155}$$

Hence a generalized eigenvector for the eigenspace $N(\boldsymbol{A} - \lambda \boldsymbol{I})$ is

$$\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{156}$$

We define the similarity trasformation $\boldsymbol{P}$ by using the eigenvectors $\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}$ and the generalized eigenvector $\boldsymbol{v}_3$ of $\boldsymbol{A}$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}. \tag{157}$$

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly independent we have that P is invertible. Clearly,

$$\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}\boldsymbol{v}_1 & \boldsymbol{A}\boldsymbol{v}_2 & \boldsymbol{A}\boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} \lambda\boldsymbol{v}_1 & \lambda\boldsymbol{v}_2 & \boldsymbol{v}_2 + \lambda\boldsymbol{v}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}}_{\boldsymbol{J}} = \boldsymbol{P}\boldsymbol{J}. \tag{158}$$

Jordan blocks. At this point it is clear that by computing the generalized eigenvectors it is always possible to construct a similarity transformation $\boldsymbol{P}$ that takes any matrix $\boldsymbol{A}$ into its Jordan canonical form

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & & & \\ & \boldsymbol{J}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{J}_p \end{bmatrix}, \tag{159}$$

[7] Note there is really no reason why we should choose $\boldsymbol{v}_1$ instead of $\boldsymbol{v}_2$ at the right hand side of (154). In fact, the choice of both eigenvectors and generalized eigenvectors is not really unique.

where p is the total number of distinct eigenvalues of $\boldsymbol{A}$. The Jordan canonical form is a block-diagonal matrix in which each block $\boldsymbol{J}_i$ can be of the form summarized in Table 1.

Matrix exponentials of Jordan blocks. The matrix exponential of the Jordan form of (159) is a block-diagonal matrix that has the matrix exponential of each Jordan block along the diagonal.

$$e^{t\boldsymbol{J}} = \begin{bmatrix} e^{t\boldsymbol{J}_1} & & & \\ & e^{t\boldsymbol{J}_2} & & \\ & & \ddots & \\ & & & e^{t\boldsymbol{J}_p} \end{bmatrix}. \tag{160}$$

Hence, to compute the matrix exponential of the Jordan form of $\boldsymbol{A}$, we just need a formula for the matrix exponential of each Jordan block in Table 1. The case in which the Jordan block is diagonal is trivial, since the matrix exponential is just the exponential of the diagonal elements. For instance,

$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix} \qquad \Rightarrow \qquad e^{t\boldsymbol{J}_i} = \begin{bmatrix} e^{t\lambda_i} & 0 \\ 0 & e^{t\lambda_i} \end{bmatrix}. \tag{161}$$

Let us now show how to compute the matrix exponential of the following Jordan blocks

$$\text{a)} \quad \boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}, \qquad \text{b)} \quad \boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}, \qquad \text{c)} \quad \boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}. \tag{162}$$

a) Let us write the 2D Jordan block as

$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}}_{\boldsymbol{B}_i} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{C}}. \tag{163}$$

The matrix commutator of $\boldsymbol{B}_i$ and $\boldsymbol{C}$ equals zero. In fact,

$$[\boldsymbol{B}_i, \boldsymbol{C}] = \boldsymbol{B}_i\boldsymbol{C} - \boldsymbol{C}\boldsymbol{B}_i = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{164}$$

This implies that[8]

$$e^{t\boldsymbol{J}_i} = e^{t(\boldsymbol{B}_i+\boldsymbol{C})} = e^{t\boldsymbol{B}_i}e^{t\boldsymbol{C}}. \tag{168}$$

Since $\boldsymbol{B}_i$ is a diagonal matrix

$$e^{t\boldsymbol{B}_i} = \begin{bmatrix} e^{t\lambda_i} & 0 \\ 0 & e^{t\lambda_i} \end{bmatrix}. \tag{169}$$

[8] In general, given two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ we have

$$e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{Z}}, \tag{165}$$

where

$$\boldsymbol{Z} = \boldsymbol{A} + \boldsymbol{B} + \frac{1}{2}[\boldsymbol{A}, \boldsymbol{B}] + \frac{1}{12}[\boldsymbol{A}, [\boldsymbol{A}, \boldsymbol{B}]] + \cdots. \tag{166}$$

This formula is known as Baker-Campbell-Hausdorff formula. If $\boldsymbol{A}$ and $\boldsymbol{B}$ commute, i.e., if $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{0}_{M_{n\times n}}$ then by (165) and (166) we have

$$e^{\boldsymbol{A}}e^{\boldsymbol{B}} = e^{\boldsymbol{A}+\boldsymbol{B}}. \tag{167}$$

Regarding the exponential of $\boldsymbol{C}$ we have the exact formula[9]

$$e^{t\boldsymbol{C}} = \boldsymbol{I} + t\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \tag{171}$$

Finally, a substitution of (171) and (169) into (168) yields the desired expression

$$e^{t\boldsymbol{J}_i} = \begin{bmatrix} e^{t\lambda_i} & 0 \\ 0 & e^{t\lambda_i} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{t\lambda_i} & te^{\lambda_i t} \\ 0 & e^{t\lambda_i} \end{bmatrix}. \tag{172}$$

b) The exponential of the 3D Jordan block

$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \tag{173}$$

can be computed using the formula (172) we just proved. In fact,

$$e^{t\boldsymbol{J}_i} = \begin{bmatrix} e^{t\lambda_i} & 0 & 0 \\ 0 & e^{t\lambda_i} & te^{\lambda_i t} \\ 0 & 0 & e^{t\lambda_i} \end{bmatrix}. \tag{174}$$

c) The exponential of the 3D Jordan block

$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \tag{175}$$

requires more work. We begin by splitting $\boldsymbol{J}_i$ as the sum of a diagonal matrix and and a non-diagonal matrix

$$\boldsymbol{J}_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & \lambda_i \end{bmatrix}}_{\boldsymbol{B}_i} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{C}}. \tag{176}$$

As before, it is straightforward to show that $\boldsymbol{B}_i$ and $\boldsymbol{C}$ commute

$$[\boldsymbol{B}_i, \boldsymbol{C}] = \boldsymbol{B}_i\boldsymbol{C} - \boldsymbol{C}\boldsymbol{B}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{177}$$

Moreover, by a direct calculation, we can show that $\boldsymbol{C}$ is again a nilpotent matrix

$$\boldsymbol{C}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \dots, \qquad \boldsymbol{C}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{178}$$

Therefore, the matrix exponential of the Jordan block (175) is

$$e^{t\boldsymbol{J}_i} = e^{t\boldsymbol{B}_i} e^{t\boldsymbol{C}} = e^{t\boldsymbol{B}_i}\left(\boldsymbol{I} + t\boldsymbol{C} + t^2\frac{\boldsymbol{C}^2}{2}\right). \tag{179}$$

[9] In fact,

$$\boldsymbol{C}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{170}$$

Of course all matrix powers $\boldsymbol{C}^k$ are all zero for $k \geq 2$ since $\boldsymbol{C}^2 = \boldsymbol{0}$, and we can write $\boldsymbol{C}^k = \boldsymbol{C}^2\boldsymbol{C}^{k-2}$. Matrices with powers that are equal to zero for some k larger than a threshold are called *nilpotent*.

Substituting (178) into (179) finally yields

$$e^{t\boldsymbol{J}_i} = \begin{bmatrix} e^{t\lambda_i} & 0 & 0 \\ 0 & e^{t\lambda_i} & 0 \\ 0 & 0 & e^{t\lambda_i} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{I}+t\boldsymbol{C}+t^2\boldsymbol{C}^2/2}. \tag{180}$$

Developing the product finally yields

$$e^{t\boldsymbol{J}_i} = \begin{bmatrix} e^{t\lambda_i} & te^{t\lambda_i} & t^2e^{t\lambda_i}/2 \\ 0 & e^{t\lambda_i} & te^{t\lambda_i} \\ 0 & 0 & e^{t\lambda_i} \end{bmatrix}. \tag{181}$$

The matrix exponential of all Jordan blocks we discussed in this section are summarized in Table 1. Formulas for matrix exponentials of higher-dimensional Jordan blocks can be computed by using the techniques we discussed in this section.

Real Jordan form. Suppose that the matrix $\boldsymbol{A}$ has two complex conjugate eigenvalues

$$\lambda_{1,2} = \alpha \pm i\beta. \tag{182}$$

The complex eigenvalues will have corresponding eigenvectors that are also complex. To obtain a real representation, we apply a change of basis that expresses these complex eigenvectors in terms of their real and imaginary parts. Specifically, by taking the real and imaginary parts of either complex eigenvector[10], we obtain two *real linearly independent vectors* that span a two-dimensional invariant subspace. The dynamics on this subspace is characterized by spirals or centers. The Jordan block associated with this transformation is given by

$$\boldsymbol{J}_{1,2} = \begin{bmatrix} \alpha & \pm\beta \\ \mp\beta & \alpha \end{bmatrix}. \tag{183}$$

The sign ambiguity in the off-diagonal entries of this block depends on the specific choice of signs for the real and imaginary parts of the eigenvectors. The exponential of such a Jordan block is

$$e^{t\boldsymbol{J}_{1,2}} = e^{\alpha t}\begin{bmatrix} \cos(\beta t) & \pm\sin(\beta t) \\ \mp\sin(\beta t) & \cos(\beta t) \end{bmatrix}. \tag{184}$$

Example: Let us compute the real Jordan form of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{185}$$

The eigenvalues are

$$\lambda_{1,2} = 1 \pm i\sqrt{3}, \qquad \lambda_3 = 1. \tag{186}$$

The eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} i\sqrt{3} \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -i\sqrt{3} \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}. \tag{187}$$

[10] Eigenvectors associated with complex conjugate eigenvalues are themselves complex conjugates.

Note that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are complex conjugates. The Jordan form corresponding to the similarity transformation $\boldsymbol{P} = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]$ is

$$\boldsymbol{J} = \begin{bmatrix} 1+i\sqrt{3} & 0 & 0 \\ 0 & 1-i\sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{188}$$

Let us now construct an alternative (real) basis for the subspace spanned by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, using the real and imaginary parts of either complex conjugate eigenvector. Let us chose $\boldsymbol{v}_1$, and define the real similarity transformation

$$\boldsymbol{P}_R = \begin{bmatrix} \mathrm{Re}(\boldsymbol{v}_1) & \mathrm{Im}(\boldsymbol{v}_1) & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ -2 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix} \tag{189}$$

the vectors $[\mathrm{Re}(\boldsymbol{v}_1)\ \ \mathrm{Im}(\boldsymbol{v}_1)]$ span the two-dimensional space with "unstable spiral" dynamics. The real Jordan form corresponding to the similarity transformation $\boldsymbol{P}_R$ is

$$\boldsymbol{J}_R = \begin{bmatrix} 1 & \sqrt{3} & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{190}$$

Note that there is an ambiguity in the sign of the off-diagonal elements. Such ambiguity depends on the sign we chose for $\mathrm{Re}(\boldsymbol{v}_1)$ and $\mathrm{Im}(\boldsymbol{v}_1)$. The exponential of the Jordan block $\boldsymbol{J}_R$ is

$$e^{t\boldsymbol{J}_R} = \begin{bmatrix} e^t\cos(\sqrt{3}t) & e^t\sin(\sqrt{3}t) & 0 \\ -e^t\sin(\sqrt{3}t) & e^t\cos(\sqrt{3}t) & 0 \\ 0 & 0 & e^t \end{bmatrix}. \tag{191}$$

Appendix C: Matrix norms compatible with vector norms

Let us define the following class of matrix norm

$$\|\boldsymbol{A}\| = \sup_{\boldsymbol{y}\neq\boldsymbol{0}_{\mathbb{R}^n}} \frac{\|\boldsymbol{A}\boldsymbol{y}\|}{\|\boldsymbol{y}\|} = \sup_{\|\boldsymbol{y}\|=1} \|\boldsymbol{A}\boldsymbol{y}\|, \tag{192}$$

where $\|\boldsymbol{A}\boldsymbol{y}\|$ and $\|\boldsymbol{y}\|$ are vector norms. Clearly, $\|\boldsymbol{A}\|$ is matrix norm, i.e., it satisfies the basic properties of a norm

- $\|\boldsymbol{A}\| \geq 0 \qquad (\|\boldsymbol{A}\| = 0$ if and only if $\boldsymbol{A} = \boldsymbol{0})$
- $\|c\boldsymbol{A}\| = |c|\,\|\boldsymbol{A}\|$ for all $c \in \mathbb{R}$
- $\|\boldsymbol{A}+\boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$

Moreover, $\|\boldsymbol{A}\|$ satisfies, by definition, the inequalities

$$\|\boldsymbol{A}\| \geq \frac{\|\boldsymbol{A}\boldsymbol{y}\|}{\|\boldsymbol{y}\|} \qquad \text{i.e.} \qquad \|\boldsymbol{A}\boldsymbol{y}\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{y}\|. \tag{193}$$

It is straightforward to show that

$$\|\boldsymbol{A}\|_\infty = \max_{i=1,..,n}\left(\sum_{j=1}^n |A_{ij}|\right), \tag{194}$$

$$\|\boldsymbol{A}\|_1 = \max_{j=1,..,n}\left(\sum_{i=1}^n |A_{ij}|\right), \tag{195}$$

$$\|\boldsymbol{A}\|_2 = \sqrt{\lambda_{\max}\left(\boldsymbol{A}^T\boldsymbol{A}\right)} = \sigma_{\max}(\boldsymbol{A}), \tag{196}$$

where $\sigma_{\max}(\boldsymbol{A})$ is the largest singular value of the matrix $\boldsymbol{A}$. For example,

$$\|\boldsymbol{A}\boldsymbol{y}\|_\infty = \max_{i=1,\dots,n}\left|\sum_{j=1}^n A_{ij}y_j\right| \leq \max_{i=1,\dots,n}\left(\sum_{j=1}^n |A_{ij}|\,|y_j|\right) \leq \|\boldsymbol{y}\|_\infty \max_{i=1,\dots,n}\left(\sum_{j=1}^n |A_{ij}|\right) \tag{197}$$

which implies that

$$\frac{\|\boldsymbol{A}\boldsymbol{y}\|_\infty}{\|\boldsymbol{y}\|_\infty} \leq \max_{i=1,\dots,n}\left(\sum_{j=1}^n |A_{ij}|\right) \qquad \text{for all } \boldsymbol{y} \neq \boldsymbol{0}_{\mathbb{R}^n}, \tag{198}$$

i.e.,

$$\|\boldsymbol{A}\|_\infty = \sup_{\boldsymbol{y}\neq\boldsymbol{0}_{\mathbb{R}^n}} \frac{\|\boldsymbol{A}\boldsymbol{y}\|_\infty}{\|\boldsymbol{y}\|_\infty} = \max_{i=1,\dots,n}\left(\sum_{j=1}^n |A_{ij}|\right). \tag{199}$$

With any compatible matrix norm available we immediately see that the function $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{y}$ is Lipschitz continuous in $\mathbb{R}^n$. In fact, we have

$$\|\boldsymbol{A}\boldsymbol{y}_1 - \boldsymbol{A}\boldsymbol{y}_2\| \leq \|\boldsymbol{A}\|\,\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \qquad \text{for all} \qquad \boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^n, \tag{200}$$

where $L = \|\boldsymbol{A}\|$ is a Lipschitz constant.

Appendix D: Solution of a linear system in terms of the matrix exponential

We first write the ODE (1) as a linear integral equation

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}(s, \boldsymbol{x}_0)ds.$$

To solve this equation we use the Picard iteration method, which is a fixed point iteration method. To this end, we define the iterative sequence

$$\boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}^{(n-1)}(s, \boldsymbol{x}_0)ds \qquad \boldsymbol{X}^{(0)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 \tag{201}$$

Picard's iterations are convergent within a temporal interval $t \in [0, T]$, where T depends on the norm of $\boldsymbol{A}$. Let us we start with $n = 1$

$$\boldsymbol{X}^{(1)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{x}_0 ds = \boldsymbol{x}_0 + \boldsymbol{A}\boldsymbol{x}_0 t = (\boldsymbol{I} + \boldsymbol{A}t)\boldsymbol{x}_0.$$

We can use this to compute $n = 2$ which gives

$$\boldsymbol{X}^{(2)}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}^{(1)}(s, \boldsymbol{x}_0)ds = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}(\boldsymbol{I} + \boldsymbol{A}t)\boldsymbol{x}_0 ds = \left(\boldsymbol{I} + \boldsymbol{A}t + \frac{t^2}{2}\boldsymbol{A}^2\right)\boldsymbol{x}_0.$$

By induction it is straightforward to show that

$$\boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0) = \left(\sum_{k=0}^n \frac{\boldsymbol{A}^k t^k}{k!}\right)\boldsymbol{x}_0.$$

Clearly,

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \lim_{n\to\infty} \boldsymbol{X}^{(n)}(t, \boldsymbol{x}_0) = \left(\lim_{n\to\infty}\sum_{k=0}^n \frac{\boldsymbol{A}^k t^k}{k!}\right)\boldsymbol{x}_0 = e^{t\boldsymbol{A}}\boldsymbol{x}_0. \tag{202}$$

Convergence of Picard's iterations. This can be easily established within the context of *Banach fixed point theorem* (or Banach contraction mapping theorem) for continuous functions, which we recall hereafter.

Theorem 5 (Banach fixed point theorem)**.** Let $T > 0$ be fixed, and let $U = C([0, T]; \mathbb{R}^n)$ denote the space of all continuous functions from $[0, T]$ to $\mathbb{R}^n$, equipped with the supremum norm

$$\|\boldsymbol{x}(t)\|_\infty = \sup_{t\in[0,T]} \|\boldsymbol{x}(t)\|, \tag{203}$$

where $\|\cdot\|$ is any vector norm on $\mathbb{R}^n$. Suppose $\mathcal{L} : U \to U$ is a linear mapping[11] for which there exists a constant $0 < M < 1$ such that

$$\|\mathcal{L}\boldsymbol{x}_1 - \mathcal{L}\boldsymbol{x}_2\|_\infty \leq M\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_\infty \quad \text{for all } \boldsymbol{x}_1, \boldsymbol{x}_2 \in U.$$

Then

1. There exists a unique fixed point $\boldsymbol{x} \in U$ such that $\mathcal{L}\boldsymbol{x}(t) = \boldsymbol{x}(t)$.
2. For any initial guess $\boldsymbol{x}^{(0)} \in U$, the Picard iteration
$$\boldsymbol{x}^{(k+1)}(t) = \mathcal{L}\boldsymbol{x}^{(k)}(t), \quad k = 0, 1, 2, \ldots,$$
converges uniformly to the unique fixed point $\boldsymbol{x}$.
3. Convergence is geometric
$$\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\|_\infty \leq \frac{M^k}{1 - M}\|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\|_\infty.$$

To show how Theorem 5 can be applied to prove convergence of Picard's iterations, define the linear operator[12] $\mathcal{L} : U \mapsto U$ as

$$\mathcal{L}\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \boldsymbol{A}\boldsymbol{x}(s)ds \qquad \forall t \in [0, T]. \tag{204}$$

We want to find conditions on the integration time T such that $\mathcal{L}$ is a (strong) contraction, i.e.,

$$\|\mathcal{L}\boldsymbol{x}_1(t) - \mathcal{L}\boldsymbol{x}_2(t)\| \leq M\|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)\| \qquad \text{for all} \qquad \boldsymbol{x}_1(t), \boldsymbol{x}_2(t) \in C([0, T]; \mathbb{R}^n) \qquad 0 < M < 1. \tag{205}$$

Using the norm defined in (203) we have

$$\begin{aligned}\|\mathcal{L}\boldsymbol{x}_1(t) - \mathcal{L}\boldsymbol{x}_2(t)\|_\infty &= \left\|\int_0^t \boldsymbol{A}(\boldsymbol{x}_1(s) - \boldsymbol{x}_2(s))\, ds\right\|_\infty \\ &\leq \int_0^t \|\boldsymbol{A}(\boldsymbol{x}_1(s) - \boldsymbol{x}_2(s))\|_\infty\, ds \\ &\leq t\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_\infty\|\boldsymbol{A}\|_\infty,\end{aligned} \tag{206}$$

where $\|\boldsymbol{A}\|_\infty$ is the matrix norm (194). Hence, the operator $\mathcal{L}$ is a contraction on $C([0, T]; \mathbb{R}^n)$ provided that

$$t\|\boldsymbol{A}\|_\infty < 1 \quad \text{for all } t \in [0, T] \quad \text{i.e.} \quad T < \frac{1}{\|\boldsymbol{A}\|_\infty}. \tag{207}$$

For any such T, the *Banach fixed-point theorem* allows us to conclude that there exists a unique fixed point of the operator $\mathcal{L}$ in the space $C([0, T]; \mathbb{R}^n)$, satisfying

$$\boldsymbol{x}(t) = \mathcal{L}\boldsymbol{x}(t), \tag{208}$$

[11]Banach fixed point theorem works for both linear and nonlinear mappings.

[12]Note that the operator $\mathcal{L}$ in equation (204) does not require $\boldsymbol{x}(t)$ to be continuously differentiable in time. Therefore, a solution to the integral equation $\mathcal{L}\boldsymbol{x}(t) = \boldsymbol{x}(t)$ is, in general, a weaker notion than a solution to the differential equation $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x}$, since the former requires only continuity of $\boldsymbol{x}(t)$, whereas the latter requires continuous differentiability.

that is,

$$\boldsymbol{X}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}\boldsymbol{X}(s, \boldsymbol{x}_0)\, ds, \qquad \forall t \in [0, T]. \tag{209}$$

The fixed point is explicitly given by (202), for all $t \in [0, T]$

Furthermore, using the semigroup property of the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$, we can apply the same argument starting from any later time $t_1 \in [0, T]$, treating $\boldsymbol{X}(t_1, \boldsymbol{x}_0)$ as the new initial condition, and integrate forward by an additional T units. In this way, the solution can be extended to cover any desired time interval.

Remark: The proof given above holds for functions that are only continuous in time. An equivalent proof can be given for functions that are continuously differentiable in time. To this end, we can replace the norm in (203) with one that also accounts for the temporal derivative of $\boldsymbol{x}(t)$ and seek conditions on t that ensure the contraction property mentioned in Theorem 5 holds with respect to this new norm, allowing us to invoke the Banach fixed-point theorem.

Remark: The Picard iteration method combined with the Banach fixed point theorem can be used to prove existence and uniqueness of the solution to systems n-dimensional nonlinear ODEs with Lipshitz continuous $\boldsymbol{f}(\boldsymbol{x})$. The sequence of steps is exactly the same as above, with $\|\boldsymbol{A}\|_\infty$ replaced by the Lipschitz constant of $\boldsymbol{f}$.