**Stability analysis of equilibria in nonlinear systems**

Consider the $n$-dimensional nonlinear dynamical system

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{1}$$

where $\boldsymbol{x}(t) = [x_1(t) \cdots x_n(t)]^T$ is a vector of phase variables, $\boldsymbol{f} : D \to \mathbb{R}^n$, and $D$ is a subset of $\mathbb{R}^n$. In this course note we study the behavior of the nonlinear system (1) in a neighborhood of a fixed point. As is well known, fixed points are solutions to the nonlinear system of algebraic equations

$$\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}. \tag{2}$$

To study the flow in a neighborhood of a fixed point $\boldsymbol{x}^*$ we consider a local coordinate system centered at $\boldsymbol{x}^*$, i.e. we define the new phase variables

$$\boldsymbol{\eta}(t, \boldsymbol{x}_0) = \boldsymbol{X}(t, \boldsymbol{x}_0) - \boldsymbol{x}^*. \tag{3}$$

Assuming that the initial condition $\boldsymbol{x}_0$ is sufficiently close to $\boldsymbol{x}^*$ and that $\boldsymbol{f}$ is sufficiently smooth, we expand

$$\boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) = \boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\eta}(t, \boldsymbol{x}_0)) \tag{4}$$

in a neighborhood of $\boldsymbol{x}^*$, i.e., for small $\boldsymbol{\eta}(t, \boldsymbol{x}_0)$. This yields

$$\boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\eta}(t, \boldsymbol{x}_0)) = \underbrace{\boldsymbol{f}(\boldsymbol{x}^*)}_{=\boldsymbol{0}} + \boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{x}^*)\boldsymbol{\eta}(t, \boldsymbol{x}_0) + \boldsymbol{g}(\boldsymbol{\eta}), \tag{5}$$

where

$$\boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{x}^*) = \begin{bmatrix} \dfrac{\partial f_1(\boldsymbol{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial f_1(\boldsymbol{x}^*)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n(\boldsymbol{x}^*)}{\partial x_1} & \cdots & \dfrac{\partial f_n(\boldsymbol{x}^*)}{\partial x_n} \end{bmatrix} \tag{6}$$

is the Jacobian[1] of $\boldsymbol{f}(\boldsymbol{x})$ evaluated at the fixed point $\boldsymbol{x}^*$, and $\boldsymbol{g}(\boldsymbol{\eta})$ is the reminder of the Taylor series at $\boldsymbol{x}^*$. Of course $\boldsymbol{g}(\boldsymbol{\eta})$ depends on $\boldsymbol{x}^*$. Moreover,

$$\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{J}_{\boldsymbol{g}}(\boldsymbol{x}^*) = \boldsymbol{0}. \tag{7}$$

These conditions imply that $\boldsymbol{\eta} = \boldsymbol{0}$ is indeed a fixed point, and that that $\boldsymbol{g}(\boldsymbol{\eta})$ is at least quadratic in $\boldsymbol{\eta}$. This allows us to write the nonlinear dynamical system (1) at $\boldsymbol{x}^*$ as

$$\begin{cases} \dfrac{d\boldsymbol{\eta}}{dt} = \boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{x}^*)\boldsymbol{\eta} + \boldsymbol{g}(\boldsymbol{\eta}) \\ \boldsymbol{\eta}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0 - \boldsymbol{x}^* \end{cases} \tag{8}$$

Note that (8) is completely equivalent to (1), since we retained all nonlinearities. Such nonlinerities are responsible for the slight variations in the local phase portraits displayed in Figure 1.

[1] The Jacobian of $\boldsymbol{f}(\boldsymbol{x})$ is a matrix-valued function that takes in a function $\boldsymbol{f}(\boldsymbol{x})$ and it returns a $n \times n$ matrix-valued function. The entries of such Jacobian matrix are functions. Of course, if we evaluate the Jacobian of $\boldsymbol{f}(\boldsymbol{x})$ at a specific point $\boldsymbol{x}^*$ then we obtain a matrix with real entries (provided $\boldsymbol{f}$ is real).

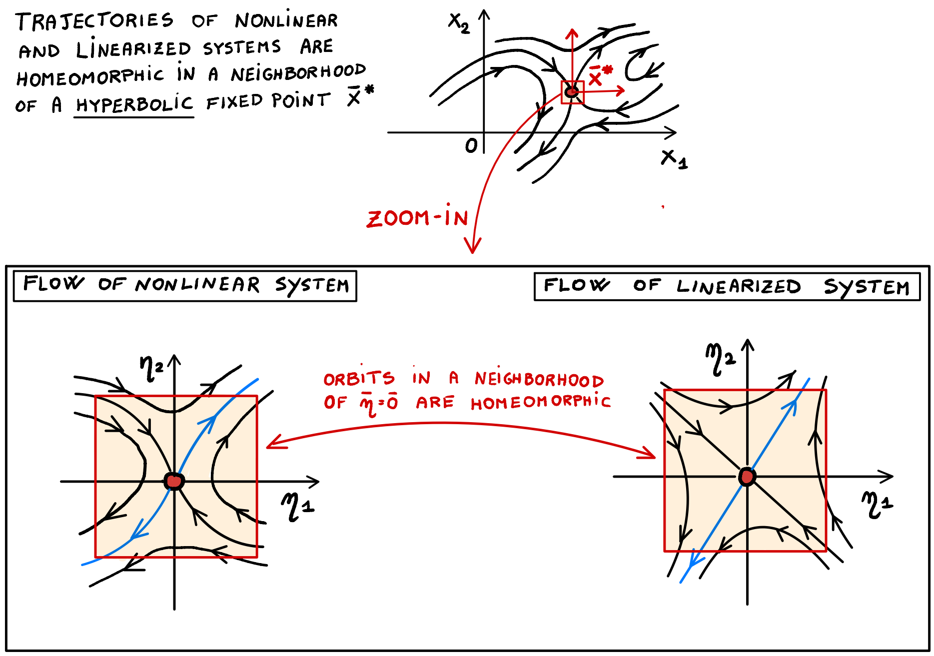



Figure 1: Geometric meaning of the Hartmman-Grobman Theorem 1. The trajectories of a nonlinear system in a neighborhood of any hyperbolic fixed point are homeomorphic to the trajectories of the linearized system at $\boldsymbol{x}^*$. This means that the trajectories of the nonlinear and linearized system are not exactly the same in the neighborhood of $\boldsymbol{x}^*$, but they can be mapped to each other by a continuous transformation that has a continuous inverse. The reason why the trajectories are not the same can be traced back to the term $\boldsymbol{g}(\boldsymbol{\eta})$ in equation (8).

**Theorem 1** (Hartman-Grobman)**.** Let $\boldsymbol{x}^* \in \mathbb{R}^n$ be a fixed point of the dynamical system (1). If the Jacobian (6) has no eigenvalue with zero real part then there exists a homeomorphism (i.e., continuous invertible mapping with continuous inverse) defined on some neighborhood of $\boldsymbol{x}^*$ that takes orbits of the linear system $\dot{\boldsymbol{\eta}} = \boldsymbol{J_f}(\boldsymbol{x}^*)\boldsymbol{\eta}$ and maps them into orbits of the system (8). The mapping preserves the orientation of the orbits.

An outline of the proof is given in L. Perko, *Differential equations and dynamical systems*, page 121.

**Remark:** Theorem 1 states that if $\boldsymbol{x}^*$ is a *hyperbolic*[2] fixed point, then the flow of the nonlinear system (1) in a neighborhood $U \subset \mathbb{R}^n$ of $\boldsymbol{x}^*$ is *homeomorphic* to the flow of the corresponding linearized system (**??**). That is, the trajectories of the nonlinear and linear systems can be mapped to each other by a continuous bijection

$$\boldsymbol{h} : U \mapsto V, \tag{9}$$

with a continuous inverse, where $V$ is the image of $U$ under $\boldsymbol{h}$. Stated mathematically, the theorem asserts that there exists a homeomorphism $\boldsymbol{h}$ such that

$$\boldsymbol{h}(\boldsymbol{X}(t,\boldsymbol{x}_0) - \boldsymbol{x}^*) = e^{t\boldsymbol{J_f}(\boldsymbol{x}^*)}\boldsymbol{h}(\boldsymbol{x}_0 - \boldsymbol{x}^*), \qquad \text{i.e.} \qquad \boldsymbol{X}(t,\boldsymbol{x}_0) = \boldsymbol{x}^* + \boldsymbol{h}^{-1}\left(e^{t\boldsymbol{J_f}(\boldsymbol{x}^*)}\boldsymbol{h}(\boldsymbol{x}_0 - \boldsymbol{x}^*)\right), \tag{10}$$

The homeomorphism maps trajectories of (8) to trajectories of $\dot{\boldsymbol{\eta}} = \boldsymbol{J_f}(\boldsymbol{x}^*)\boldsymbol{\eta}$ (without $g(\boldsymbol{\eta})$) in some neighborhood of $\mathbf{0}$.

[2] A fixed point $\boldsymbol{x}^*$ is called *hyperbolic* if the Jacobian matrix $\boldsymbol{J_f}(\boldsymbol{x}^*)$ has no eigenvalues with zero real part.

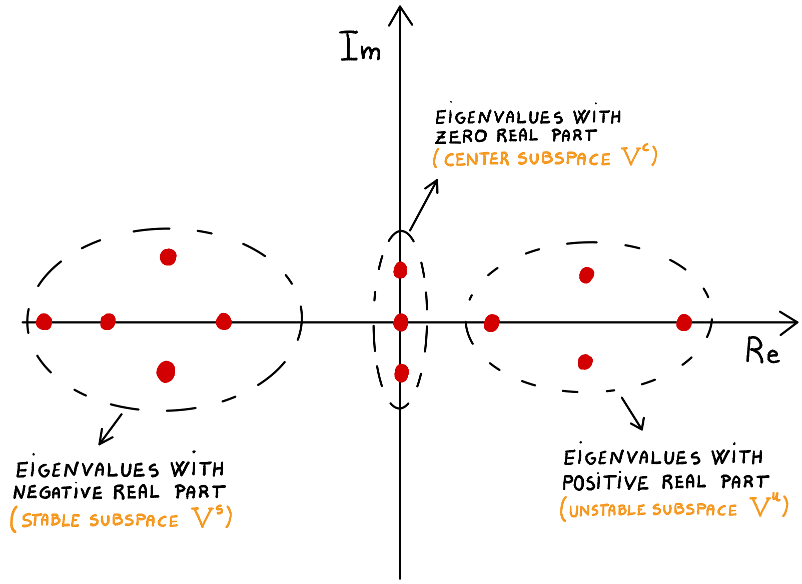



Figure 2: Eigenvalues of the Jacobian matrix $\boldsymbol{J_f}(\boldsymbol{x}^*)$, and definition of the associated subspaces.

**Stable, unstable, and center subspaces**

The eigenvalues of the Jacobian matrix $\boldsymbol{J}_f(\boldsymbol{x}^*)$ and the associated subspaces can be grouped into three main classes (see Figure 2):

- **Stable subspace.** We denote the subspace spanned by the eigenvectors and the generalized eigenvectors associated with eigenvalues with negative real part as $V^s$. The subspace $V^s$ is called *stable subspace* (or stable eigenspace if it is spanned by eigenvectors).

- **Unstable subspace.** We denote the subspace spanned by the eigenvectors and the generalized eigenvectors associated with eigenvalues with positive real part as $V^u$. The subspace $V^u$ is called *unstable subspace* (or unstable eigenspace if it is spanned by eigenvectors).

- **Center subspace.** We denote the subspace spanned by the eigenvectors and the generalized eigenvectors associated with eigenvalues with zero real part as $V^c$. The subspace $V^c$ is called *center subspace* (or center eigenspace if it is spanned by eigenvectors).

The Hartman-Grobman theorem applies to a fixed point $\boldsymbol{x}^*$ with center subspace $V^c$ reducing to just one element, i.e., $V^c = \{\mathbf{0}_{\mathbb{R}^n}\}$. This means that $\dim(V^c) = 0$, i.e., the center subspace is zero dimensional.

On the other hand, the *center manifold* theorem discussed hereafter provides useful information on the stable, unstable, and center manifolds associated to a fixed point $\boldsymbol{x}^*$.

A **manifold** in $\mathbb{R}^n$ is a subset that "locally looks like" $\mathbb{R}^k$ for some $1 \leq k \leq n$. That is, around every point, it resembles flat $k$-dimensional Euclidean space, even though globally it might be curved or embedded in a higher-dimensional space.

In other words, we can locally flatten a manifold in $\mathbb{R}^n$ using a smooth coordinate transformation. For example, a smooth (non-intersecting) curve in $\mathbb{R}^2$ is a one-dimensional manifold. Similarly, a smooth surface in $\mathbb{R}^3$ is a two-dimensional manifolds. A smooth curve in $\mathbb{R}^n$ is a one-dimensional manifold that can be parameterized as $\boldsymbol{x}(t) \in \mathbb{R}^n$. Similarity, a smooth surface in $\mathbb{R}^n$ is a two-dimensional manifold that can be parameterized as $\boldsymbol{x}(u, v) \in \mathbb{R}^n$, where $u$ and $v$ are scalar parameters.

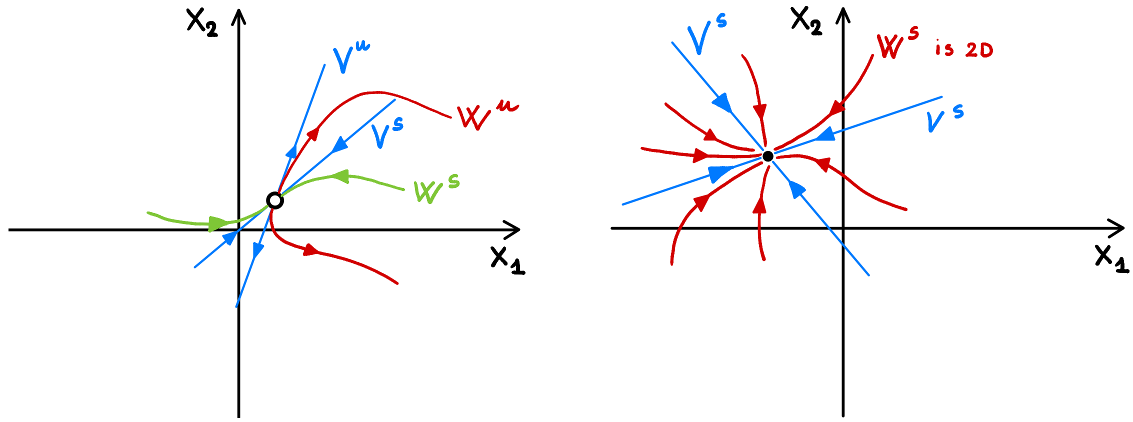



Figure 3: Stable and unstable eigenspaces $V^s$ and $V^u$, and stable and unstable manifolds $W^s$ and $W^u$ of a two-dimensional saddle node and a two-dimensional stable node. Note that the stable and unstable manifolds of the saddle node are one-dimensional and tangent to the stable and unstable eigenspaces at fixed point. The stable eigenspace of the stable node is two-dimensional. Hence the the stable manifold is two-dimensional as well. Hence the tangency condition of $W^s$ to $V^s$ in this case reduces to the trivial statement that all trajectories belong to the stable manifold, at least locally.

In the context of dynamical systems, an *invariant manifold* $W \subseteq \mathbb{R}^n$ is a manifold such that for all $\boldsymbol{x}_0 \in W$ we have that $X(t, \boldsymbol{x}_0) \in W$ for all $t$.

**Theorem 2** (Center manifold theorem)**.** Let $\boldsymbol{x}^* \in \mathbb{R}^n$ be a fixed point of the dynamical system (1), and let $V^s$, $V^u$ and $V^c$ be the stable, unstable and center subspaces defined by (generalized) eigendecomposition of the Jacobian matrix $\boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{x}^*)$ defined in (6). Then there exist two unique stable and unstable invariant manifolds $W^s$ and $W^u$ of the same dimension of $V^s$ and $V^u$ and tangential to $V^s$ and $V^c$ at $\boldsymbol{x}^*$, and a (not necessarily unique) center manifold $W^c$ of the same dimension of $V^c$ and tangential to $V^c$ at $\boldsymbol{x}^*$. If $\boldsymbol{f}$ in (1) is of class $C^k$ then $W^s$ and $W^u$ are of class $C^k$, while $W^c$ is of class $C^{k-1}$.

An invariant manifold $W \subseteq \mathbb{R}^n$ is a manifold such that for all $\boldsymbol{x}_0 \in W$ we have that $X(t, \boldsymbol{x}_0) \in W$.

It is useful to sketch the stable and unstable subspaces $V^s$ and $V^u$ together with the stable and unstable manifolds $W^s$ and $W^u$ for 2D a saddle node and for a 2D stable node. In the latter case, the stable subspace has dimension 2, and therefore all curves in a neighborhood of $\boldsymbol{x}^*$ are part of the stable manifold $W^s$.

### Stability analysis of hyperbolic fixed points in two-dimensional dynamical systems

In this section we provide a few examples of stability analysis of a hyperbolic fixed point in two-dimensional nonlinear dynamical systems.

**Example:** Consider the following Volterra-Lotka model governing the population dynamics two interacting species competing for some common resource

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(3 - x_1 - 2x_2) \\ \dfrac{dx_2}{dt} = x_2(2 - x_1 - x_2) \end{cases} \tag{11}$$

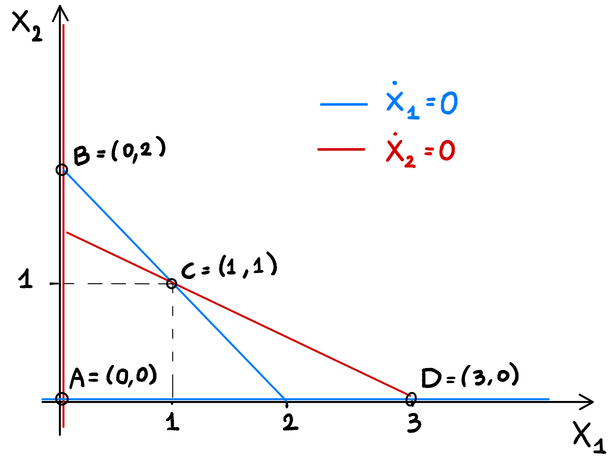



Figure 4: Fixed points of the Volterra-Lotka model (11).

The nullclines are

$$\dot{x}_1 = 0 \quad \Rightarrow \quad x_1 = 0, \qquad x_2 = \frac{3}{2} - \frac{1}{2}x_1, \tag{12}$$

$$\dot{x}_2 = 0 \quad \Rightarrow \quad x_2 = 0, \qquad x_2 = 2 - x_1. \tag{13}$$

Fixed points are located at the intersections of the nullclines. As shown in Figure 4 we obtain

$$\boldsymbol{x}_A^* = (0,0), \qquad \boldsymbol{x}_B^* = (0,2), \qquad \boldsymbol{x}_C^* = (1,1), \qquad \boldsymbol{x}_D^* = (3,0). \tag{14}$$

The Jacobian of (11) is easily obtained as

$$\boldsymbol{J}_f(\boldsymbol{x}) = \begin{bmatrix} 3 - 2x_1 - 2x_2 & -2x_1 \\ -x_2 & 2 - x_1 - 2x_2 \end{bmatrix}. \tag{15}$$

Let us study the flow of the nonlinear system in a neighborhood of the fixed point $\boldsymbol{x}_C^* = (1,1)$. The Jacobian at $\boldsymbol{x}_C^*$ is

$$\boldsymbol{J}_f(\boldsymbol{x}_C^*) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}, \tag{16}$$

and it has eigenvalues

$$\lambda_1 = -1 - \sqrt{2} < 0, \qquad \lambda_2 = -1 + \sqrt{2} > 0. \tag{17}$$

Therefore the fixed point $\boldsymbol{x}_C^*$ is hyperbolic (saddle node). The stable and unstable eigenspaces of the saddle node are spanned by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \tag{18}$$

which are eigenvectors of (16) corresponding to $\lambda_1$ and $\lambda_2$. Based on Theorem 2, the stable and unstable manifolds of the saddle node are tangent to the tangent eigenspaces stable and unstable manifolds are tangent to the eigendirections. Proceeding similarly for the other points, it is straightforward to find that $\boldsymbol{x}_A^*$ is an unstable node, while $\boldsymbol{x}_B^*$ and $\boldsymbol{x}_D^*$ are stable nodes. In Figure 5 we sketch the phase portrait of the system, and compare it with a numerically computed portrait.

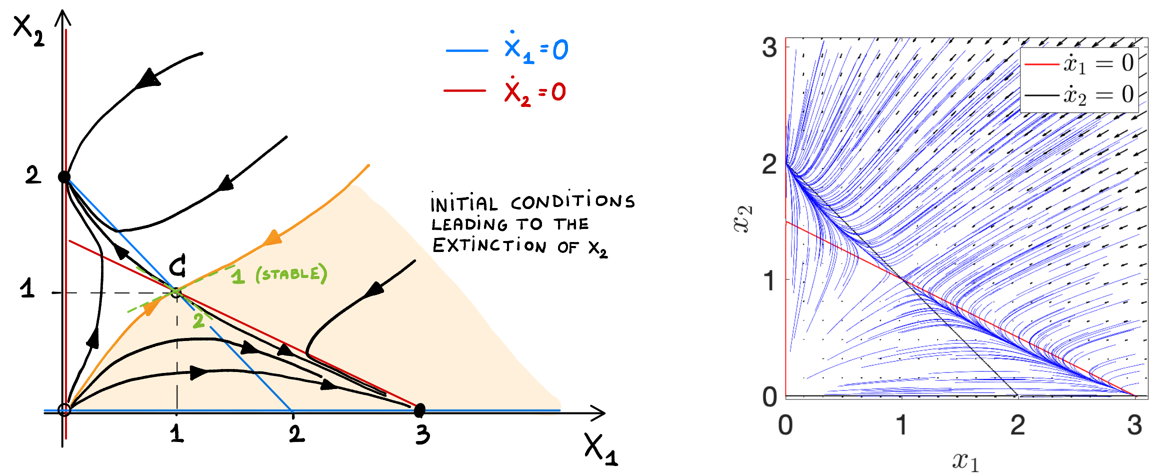



Figure 5: Phase portrait of the Volterra-Lotka model (11). The stable manifold of the saddle node determines which species is going to survive.

**Example:** Consider the nonlinear system

$$\begin{cases} \dfrac{dx_1}{dt} = 1 - (\mu+1)x_1 + x_1^2 x_2 \\[2ex] \dfrac{dx_2}{dt} = \mu x_1 - x_1^2 x_2 \end{cases} \tag{19}$$

where $\mu > 0$ is a real parameter. We allow $\mu$ to vary[3], since this will change the location of the fixed points and their stability properties. The nullclines are obtained by setting

$$\begin{cases} 1 - (\mu+1)x_1 + x_1^2 x_2 = 0 \\[2ex] x_1(\mu - x_1 x_2) = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} x_2 = \dfrac{\mu+1}{x_1} - \dfrac{1}{x_1^2} \quad (\text{for } x_1 \neq 0) \\[2ex] x_1 = 0, \quad \text{or} \quad x_2 = \dfrac{\mu}{x_1} \end{cases}$$

The fixed points are at the intersections of the nullclines. By substituting $x_2 = \mu/x_1$ into the equation defining the nullcline $\dot{x}_1 = 0$ we obtain

$$\frac{\mu}{x_1} = \frac{\mu+1}{x_1} - \frac{1}{x_1^2} \qquad \Rightarrow \qquad x_1^*(\mu) = 1. \tag{20}$$

Correspondingly,

$$\begin{aligned} x_2^*(\mu) =& \frac{\mu+1}{x_1^*(\mu)} - \frac{1}{x_1^*(\mu)^2} \\ =& \mu + 1 - 1 \\ =& \mu. \end{aligned} \tag{21}$$

Therefore, we obtain the unique fixed point

$$(x_1^*(\mu), x_2^*(\mu)) = (1, \mu). \tag{22}$$

The Jacobian of the system (19) is

$$J_{\boldsymbol{f}}(x_1, x_2, \mu) = \begin{bmatrix} -(\mu+1) + 2x_1 x_2 & x_1^2 \\ \mu - 2x_1 x_2 & -x_1^2 \end{bmatrix}. \tag{23}$$

[3]By allowing $\mu$ in (19) to vary, we are effectively studying potential bifurcations of the system, in particular bifurcations of equilibria.

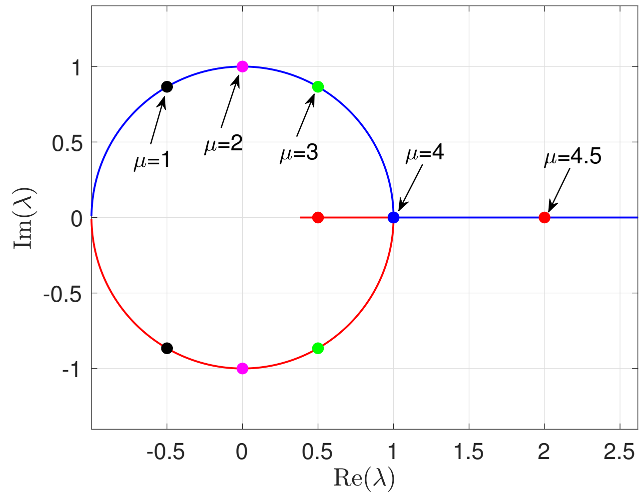



Figure 6: Eigenvalues of the Jacobian matrix (24) as a function of $\mu$.

The (linear) stability of the fixed point (22) is determined by the eigenvalues of

$$J_{\boldsymbol{f}}(x_1^*(\mu), x_2^*(\mu), \mu) = \begin{bmatrix} \mu - 1 & 1 \\ -\mu & -1 \end{bmatrix} \tag{24}$$

The associated characteristic polynomial

$$p(\lambda) = \lambda^2 - (\mu - 2)\lambda + 1 \tag{25}$$

has roots

$$\lambda_{1,2}(\mu) = \frac{(\mu - 2) \pm \sqrt{(\mu - 2)^2 - 4}}{2}. \tag{26}$$

In Figure 6 we plot the eigenvalues (26) as a function of $\mu$. Based on such eigenvalue analysis, it is seen that the fixed point (22) is:

- a stable spiral for $0 < \mu < 2$;
- a non-hyperbolic fixed point for $\mu = 2$. Center manifold analysis outlined later in this course note allows us to conclude that the non-hyperbolic fixed point is a stable spiral;
- an unstable spiral for $2 < \mu < 4$;
- an unstable degenerate node for $\mu = 4$;
- a repellor for $\mu > 4$.

For $\mu = 2$ linear stability analysis predicts a center ($\lambda_{1,2} = \pm i$). However, such fixed point is not hyperbolic and therefore such conclusion does not hold. Indeed the analysis of the center manifold outlined later in this course note allows us to conclude that for $\mu = 2$ we have a stable spiral. For $\mu = 4$ we have $\lambda_{1,2} = 1$. The geometric multiplicity of such eigenvalue is 1, and therefore at $\mu = 4$ we have an unstable degenerate node. The phase portrait of the system is shown in Figure 7 for different values of $\mu$.

### One-dimensional center manifolds in two-dimensional dynamical systems

We now examine the stability of non-hyperbolic fixed points in a two-dimensional dynamical system where one eigenvalue is zero. The local behavior near such fixed points can be analyzed by studying the dynamics

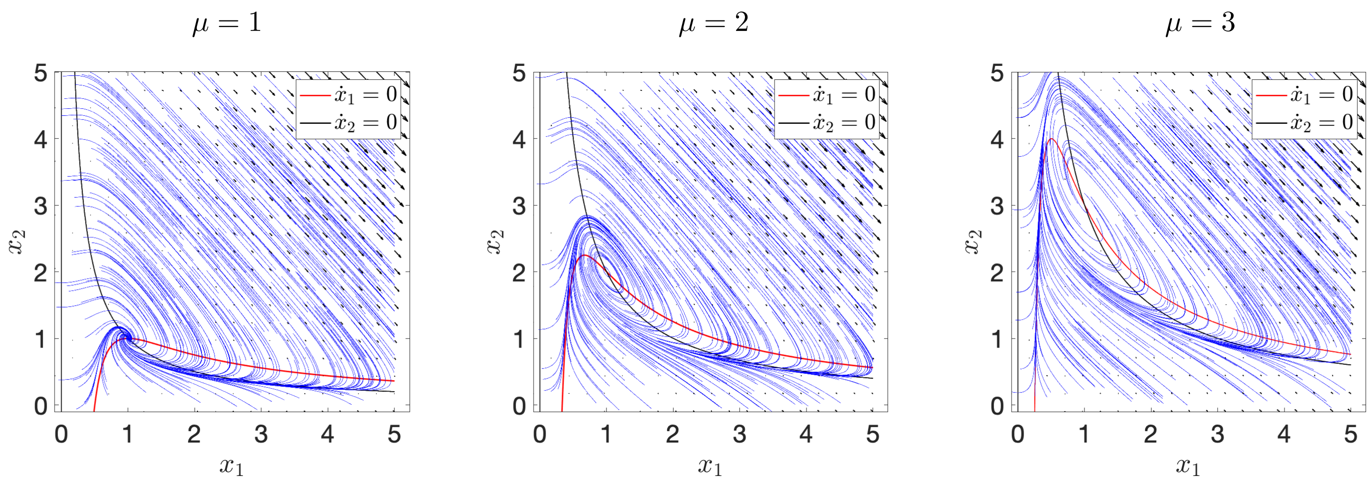



Figure 7: Phase portraits of (19) for different values of $\mu$.

restricted to the center manifold $W^c$ in a neighborhood of the equilibrium point $\boldsymbol{x}^* \in \mathbb{R}^2$. To this end, we represent such *local center manifold* $W^c$ as a graph of a smooth function $h$, i.e.,

$$W^c = \{(x_1, x_2) \in \mathbb{R}^2 \quad \text{such that} \quad x_2 = h(x_1) \quad \text{for all } x_1 \text{ in a neighborhood of } x_1^*\}. \tag{27}$$

According to the center manifold Theorem 2, there are three conditions that the function $h(x_1)$ needs to satisfy in order to represent the center manifold in a neighborhood of the fixed point $\boldsymbol{x}^*$:

1. $(x_1, h(x_1))$ needs to pass through the fixed point, i.e.,
$$x_2^* = h(x_1^*) \tag{28}$$

2. $h(x_1)$ needs to be tangent to $V^c$ at the fixed point $\boldsymbol{x}^*$. This means that the slope $h(x_1)$ must be the same as the slope[4] of $V^c$ at $x_1^*$. Such slope is identified by the "center" eigenvector of $\boldsymbol{J_f}(\boldsymbol{x}^*)$.

3. $W^c$ must be an invariant manifold. This means that any trajectory trajectory $(x_1(t), x_2(t))$ on $W^c$ must satisfy
$$x_2(t) = h(x_1(t)) \quad \Rightarrow \quad \frac{dx_2}{dt} = \frac{dh(x_1)}{dx_1}\frac{dx_1}{dt}, \tag{29}$$
i.e.,
$$f_2(x_1, h(x_1)) = \frac{dh(x_1)}{dx_1} f_1(x_1, h(x_1)). \tag{30}$$

These three conditions allow us to determine a power series expansion of the (one-dimensional) center manifold $W^c$ in a neighborhood of the fixed point $\boldsymbol{x}^*$. Let's see some examples.

**Example:** Consider the nonlinear system

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 x_2 \\ \dfrac{dx_2}{dt} = -x_2 - x_1^2 \end{cases} \tag{31}$$

[4]If the center subspace $V^s$ is a vertical line then we need to compute a preliminary coordinate transformation, e.g., use the so-called normal coordinates.

The nullclines are

$$\dot{x}_1 = 0 \quad \Leftrightarrow \quad x_1 = 0 \quad \text{or} \quad x_2 = 0, \tag{32}$$
$$\dot{x}_2 = 0 \quad \Leftrightarrow \quad x_2 = -x_1^2. \tag{33}$$

Hence, there exists only one fixed point at the intersection of the nullclines which is

$$\boldsymbol{x}^* = (0,0). \tag{34}$$

The Jacobian of the system (31) is

$$\boldsymbol{J_f}(\boldsymbol{x}) = \begin{bmatrix} x_2 & x_1 \\ -2x_1 & -1 \end{bmatrix}. \tag{35}$$

By evaluating $\boldsymbol{J_f}(\boldsymbol{x})$ at the fixed point $\boldsymbol{x}^* = (0,0)$ we obtain

$$\boldsymbol{J_f}(\boldsymbol{0}) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{36}$$

The eigenvalues of $\boldsymbol{J_f}(\boldsymbol{0})$ are

$$\lambda_c = 0 \quad \text{and} \quad \lambda_s = -1. \tag{37}$$

Correspondingly, we have a center eigenspace $V^c$ and a stable eigenspace $V^s$, both of dimension one. Such eigenspaces are spanned by the eigenvectors

$$\boldsymbol{v}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{38}$$

In Figure 8 we sketch the nullclines and the eigenspaces $V^c$ and $V^s$. Next, we compute the local center

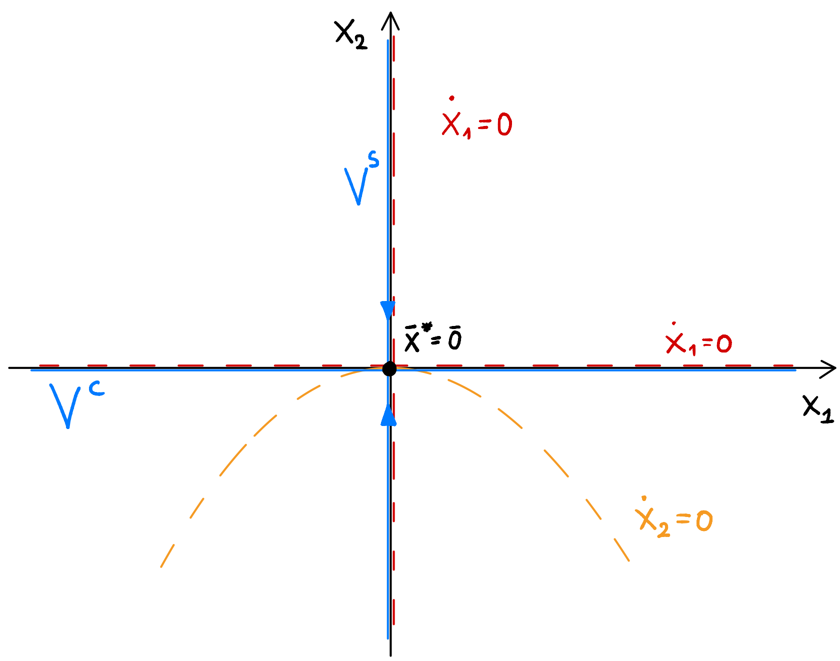



Figure 8: Nonlinear system (31). Stable ($V^s$) and center ($V^c$) eigenspaces associated with the fixed point $\boldsymbol{x}^* = (0,0)$.

manifold $W^c$ in a neighborhood of the fixed point $\boldsymbol{x}^* = (0,0)$. To this end, we consider the following power series expansion of the function $h(x_1)$ appearing in (27)

$$x_2 = h(x_1) = a + bx_1 + cx_1^2 + dx_1^3 + \cdots, \tag{39}$$

where $a$, $b$, $c$, etc. are coefficients to be determined. By enforcing that $W^c$ passes through the fixed point $(0,0)$ and is tangent to $V^c$ at $(0,0)$ we obtain

$$\begin{cases} 0 = h(0) = a \quad \Leftrightarrow \quad a = 0 \\ \\ 0 = h'(0) = b \quad \Leftrightarrow \quad b = 0 \end{cases} \tag{40}$$

Therefore we are left with

$$h(x_1) = cx_1^2 + dx_1^3 + ex_1^4 + \cdots \tag{41}$$

At this point we impose that the dynamics on the local center manifold $W^c$ is invariant, which means that any trajectory with initial condition on $W^c$ stays on $W^c$. This condition is expressed mathematically by equation (30), which can written the system (31) as

$$-h(x_1) - x_1^2 = \underbrace{\left(2cx_1 + 3dx_1^2 + \cdots\right)}_{h'(x_1)} x_1 h(x_1). \tag{42}$$

Substituting $h(x_1)$ yields

$$-\left(cx_1^2 + dx_1^3 + ex_1^4 \cdots\right) - x_1^2 = \left(2cx_1 + 3dx_1^2 + \cdots\right) x_1 \left(cx_1^2 + dx_1^3 + \cdots\right), \tag{43}$$

i.e.,

$$-(c+1)\, x_1^2 - dx_1^3 - ex_1^4 + \cdots = 2c^2 x_1^4 + 5cdx_1^5 + \cdots . \tag{44}$$

Since we are free to choose $x_1$ as small as we like, the previous equation yields the following conditions (match the coefficients multiplying the same power of $x_1$ at the left and the right hand sides)

$$c + 1 = 0, \qquad d = 0, \qquad -e = 2c^2, \tag{45}$$

i.e.,

$$c = -1, \qquad d = 0, \qquad e = -2. \tag{46}$$

This yields the following power series expansion of the local center manifold $W^c$

$$x_2 = h(x_1) = -x_1^2 - 2x_1^4 + \cdots . \tag{47}$$

The dynamics on this manifold can be obtained by substituting $x_2 = h(x_1)$ into the first equation of the system (31). This yields

$$\frac{dx_1}{dt} = -x_1^3 - 2x_1^5 + \cdots \tag{48}$$

Hence $\dot{x}_1$ always points towards the origin when evaluated along the manifold $W^c$, i.e., $W^c$ is *stable* (see Figure 9). In Figure 10 we plot the phase portrait of (31) computed numerically.

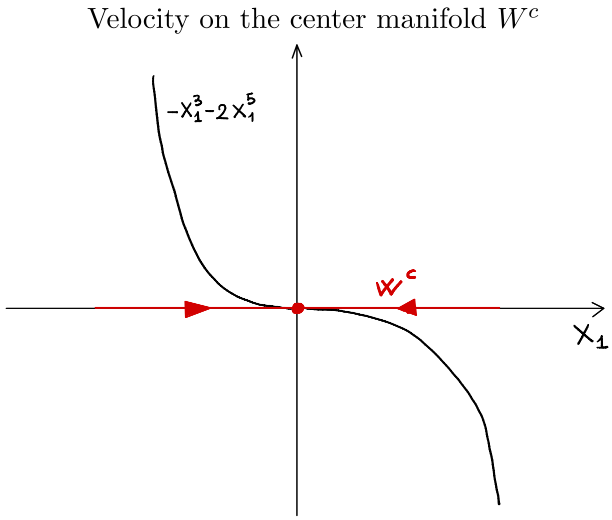
**Example:** Let us provide another example of analysis of a two-dimensional non-hyperbolic fixed point. To this end, consider the nonlinear system

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 x_2 \\ \dfrac{dx_2}{dt} = x_1 - x_2 \end{cases} \tag{49}$$

The nullclines are

$$\dot{x}_1 = 0 \quad \Leftrightarrow \quad x_1 = 0 \quad \text{or} \quad x_2 = 0, \tag{50}$$

$$\dot{x}_2 = 0 \quad \Leftrightarrow \quad x_2 = x_1. \tag{51}$$



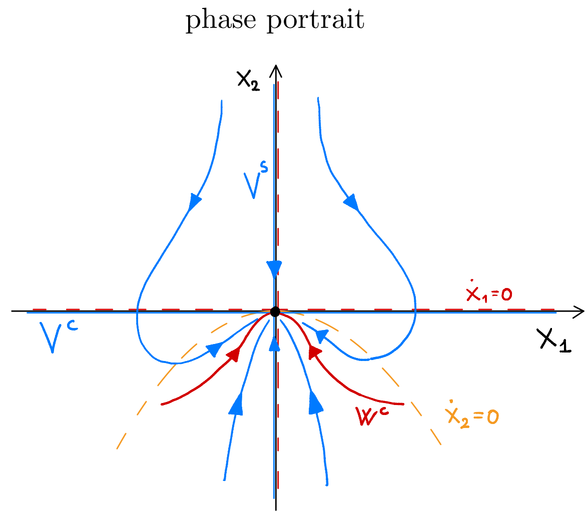




Figure 9: Nonlinear system (31). Local center manifold $W^c$ at the non-hyperbolic fixed point $(0,0)$.

Hence, there exists only one fixed point at

$$\boldsymbol{x}^* = (0,0). \tag{52}$$

The Jacobian of the system (49) is

$$\boldsymbol{J_f}(\boldsymbol{x}) = \begin{bmatrix} -x_2 & -x_1 \\ 1 & -1 \end{bmatrix} \tag{53}$$

By evaluating $\boldsymbol{J_f}(\boldsymbol{x})$ at the fixed point $\boldsymbol{x}^* = (0,0)$ we obtain

$$\boldsymbol{J_f}(\boldsymbol{0}) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}. \tag{54}$$

The eigenvalues of $\boldsymbol{J_\mathbf{f}}(\boldsymbol{0})$ are

$$\lambda_c = 0 \quad \text{and} \quad \lambda_s = -1. \tag{55}$$

Correspondingly we have a center eigenspace $V^c$ and a stable eigenspace $V^s$, both of dimension one. Such eigenspaces are spanned by the eigenvectors

$$\boldsymbol{v}_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{56}$$

To study stability of the non-hyperbolic fixed point $\boldsymbol{x}^* = (0,0)$, we compute the local center manifold $W^c$ at $\boldsymbol{x}^*$. Based on Theorem 2, $W^c$ is a $C^\infty$ one-dimensional manifold and therefore it can be represented locally as a graph of a $C^\infty$ one-dimensional function $h$ as

$$x_2 = h(x_1). \tag{57}$$

The function $h$ must satisfies the conditions

$$\begin{cases} h(0) = 0 & W^c \text{ passes through the fixed point } \boldsymbol{x}^* = (0,0), \\ h'(0) = 1 & W^c \text{ is tangent to } V^c \text{ at the fixed point } \boldsymbol{x}^* = (0,0). \end{cases} \tag{58}$$

Expanding $h(x_1)$ in a power series at $\boldsymbol{x}^* = (0,0)$ yields

$$h(x_1) = a + bx_1 + cx_1^2 + dx_1^3 + \cdots. \tag{59}$$

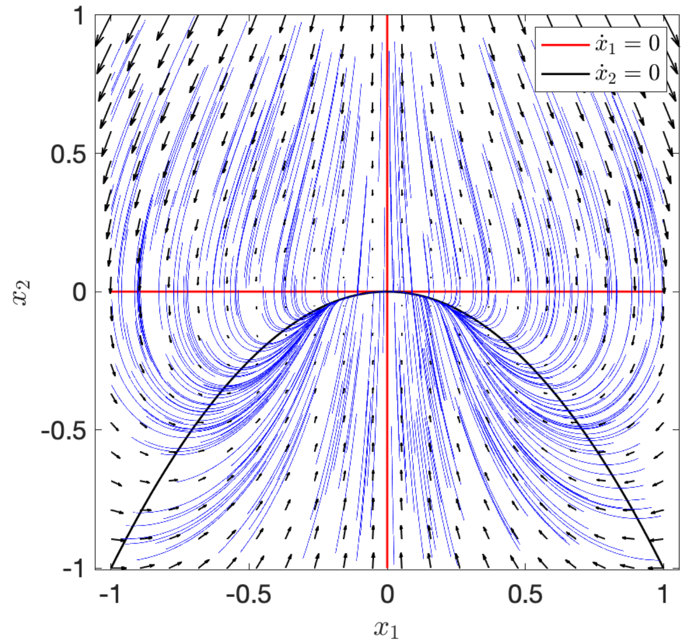



Figure 10: Phase portrait of the dynamical system (31). Note that the numerical results indicate that there may be an infinite number of center manifolds at $\boldsymbol{x}^* = (0, 0)$ (all curves passing through $(0, 0)$ with horizontal tangent at $(0, 0)$). However, the Taylor series expansions of any two center manifolds at $(0, 0)$ agree to all orders.

By enforcing conditions (58) we obtain

$$a = 0, \qquad b = 1. \tag{60}$$

Hence,

$$h(x_1) = x_1 + cx_1^2 + dx_1^3 + \cdots . \tag{61}$$

As before, the other coefficients can be obtained by imposing that $W^c$ is an invariant manifold, i.e., that trajectories starting in $W^c$ stay in $W^c$. This is equivalent to imposing that the dynamical system (49) has (57) as trajectory, i.e.,

$$x_2(t) = h(x_1(t)) \qquad \text{for all } t \geq 0, \tag{62}$$

where $(x_1(t), x_2(t))$ is a solution of (49). Differentiating (62) with respect to time yields and using (49) yields

$$x_1 - h(x_1) = -\frac{dh(x_1)}{dx_1} x_1 h(x_1). \tag{63}$$

Substituting the power series (61) into the previous equation we obtain

$$x_1 - x_1 - cx_1^2 - dx_1^3 - \cdots = -x_1 \left(1 + 2cx_1 + 3dx_1^2 + \cdots .\right) \left(x_1 + cx_1^2 + dx_1^3 + \cdots\right), \tag{64}$$

i.e.,

$$-cx_1^2 - dx_1^3 - \cdots = -x_1^2 - 3cx_1^3 + \cdots \quad \Rightarrow \quad c = 1, \quad d = 3. \tag{65}$$

Hence, the power series expansion of the center manifold $W^c$ in a neighborhood of $\boldsymbol{x}^* = (0, 0)$ is

$$x_2 = h(x_1) = x_1 + x_1^2 + 3x_1^3 + \cdots . \tag{66}$$

The dynamics on the manifold $W^c$ is obtained by substituting (62) into (49). This yields

$$\dot{x}_1 = -x_1(x_1 + x_1^2 + 3x_1^3 + \cdots) = -x_1^2 - x_1^3 - 3x_1^4 + \cdots . \tag{67}$$

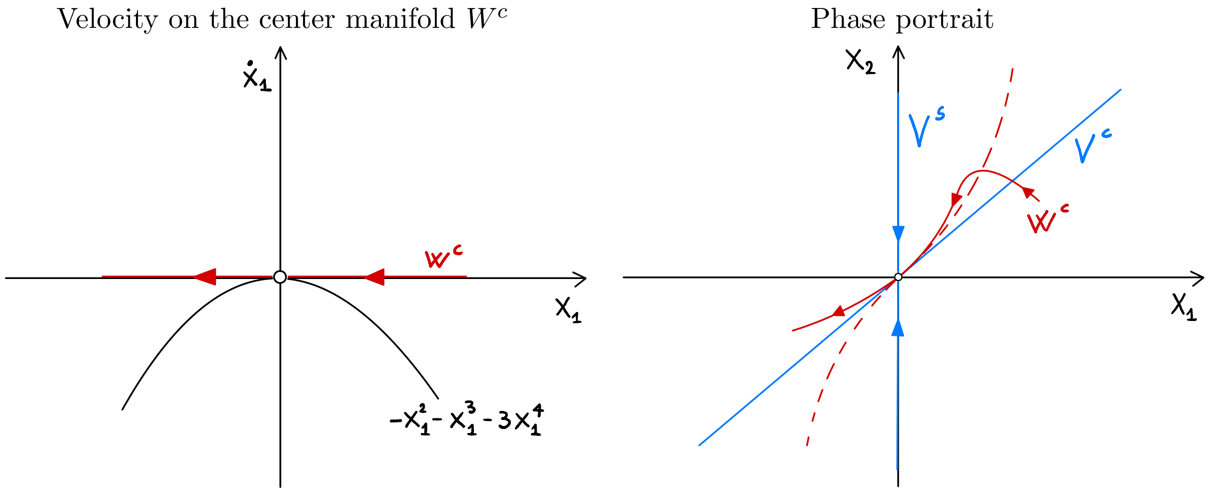



Figure 11: Nonlinear system (49). Stable and center eigenspaces $V^s$ and $V^c$, and local center manifold $W^c$ at the non-hyperbolic fixed point $(0, 0)$.

The right hand side suggests of this equation that the $x_1$ component of the velocity on the center manifold $W^c$ always points left (see Figure 11). Hence the fixed point $(0, 0)$ is unstable. In Figure 12 we plot the phase portrait of (31) computed numerically.

**Non-uniqueness of center manifolds.** We've mentioned in Theorem 2 that center manifolds need not be unique. Specifically if $\boldsymbol{f}(\boldsymbol{x})$ is $C^\infty$ then it is possible to find a $C^r$ center manifold for each $r < \infty$. This can be seen from the following simple example. Consider the dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = x_1^2 \\ \dfrac{dx_2}{dt} = -x_2 \end{cases} \tag{68}$$

clearly, $(x_1, x_2) = (0, 0)$ is a fixed point. The stable manifold $W^s$ is the vertical axis $x_1 = 0$. Moreover, $x_2 = 0$ is an invariant center manifold, but there are other center manifolds. In fact, eliminating $t$ as the independent variable in (68), we obtain (for $x_1 \neq 0$)

$$\frac{dx_2}{dx_1} = -\frac{x_2}{x_1^2} \qquad \Rightarrow \qquad x_2(x_1) = \beta e^{1/x_1} \qquad \beta \in \mathbb{R}. \tag{69}$$

Thus, the curves given by

$$h(x_1) = \begin{cases} \beta e^{1/x_1} & x_1 < 0 \\ 0 & x_1 \geq 0 \end{cases} \tag{70}$$

are a one-parameter (parametrized by $\beta$) family of center manifolds of $(x_1, x_2) = (0, 0)$. These center manifolds are shown in Figure 13. It is easy to verify indeed that $x_2(t) = \beta e^{1/x_1(t)}$ is an invariant manifold for the system (68). Moreover it is tangent to $V^c$ ($x_1$ axis), and it passes through $(0, 0)$ (for $x_1 \to 0^-$). This example brings up the following question:

- *In approximating the local center manifold via power series expansions, which center manifold is actually being approximated?*

It can be shown that any two center manifolds of a given fixed point differ by (at most) transcendentally small terms. Thus, the Taylor series expansions of any two center manifolds at a given fixed point agree to all orders. Moreover, it can be shown that for an analytical system, if the series expansion of $h$ converges, then there exists a unique analytical center manifold. The function (70) is of class $C^\infty$ but is not analytic at 0 (not convergent power series expansion at $x_1 = 0$).

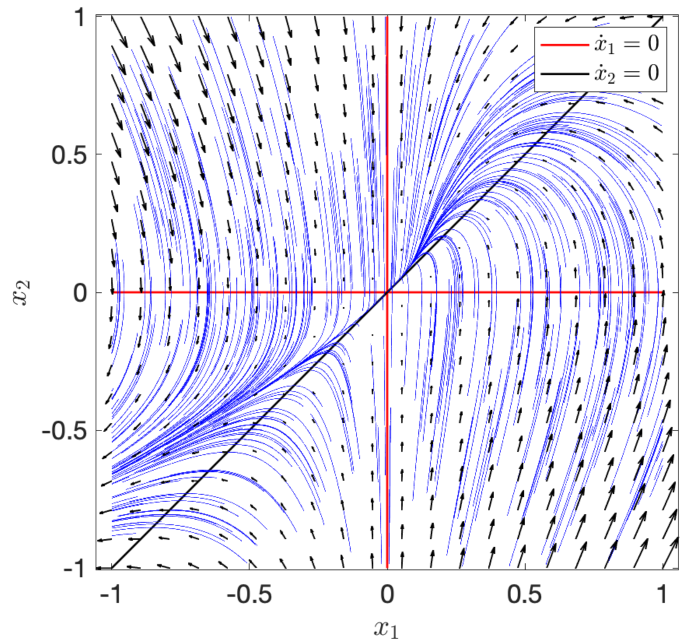



Figure 12: Phase portrait of the dynamical system (49).

**Two-dimensional center manifolds for imaginary eigenvalues**

Let us consider the case where the Jacobian matrix $\boldsymbol{J_f}(\boldsymbol{x}^*)$ in (8) has two imaginary (complex conjugate) eigenvalues, i.e.,

$$\lambda_1 = i\omega \qquad \lambda_2 = -i\omega, \tag{71}$$

where $\omega$ is a nonzero real number. In Appendix A we show that the real Jordan form of $\boldsymbol{J}_f(\boldsymbol{x}^*)$ is

$$\boldsymbol{A} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}. \tag{72}$$

Such real Jordan form is obtained by a real similarity transformation $\boldsymbol{P}$ that has the real and the imaginary part of one eigenvector as columns. By defining new variables

$$\boldsymbol{q} = \boldsymbol{P}^{-1}\boldsymbol{\eta} \tag{73}$$

it is straightforward to transform the dynamical system (8) to

$$\begin{cases} \dfrac{dq_1}{dt} = \omega q_2 + H_1(q_1, q_2) \\ \dfrac{dq_2}{dt} = -\omega q_1 + H_2(q_1, q_2) \end{cases} \tag{74}$$

It is important to emphasize that the coordinate system we use in (8) is centered at the fixed point $\boldsymbol{x}^*$. Correspondingly the transformation induced by the similarity transformation $\boldsymbol{P}$ that yields the real Jordan form of $\boldsymbol{J_f}$ transforms the coordinate system $\boldsymbol{\eta}$ centered at $\boldsymbol{x}^*$ to another coordinate system $\boldsymbol{q}$ still centered at $\boldsymbol{x}^*$.

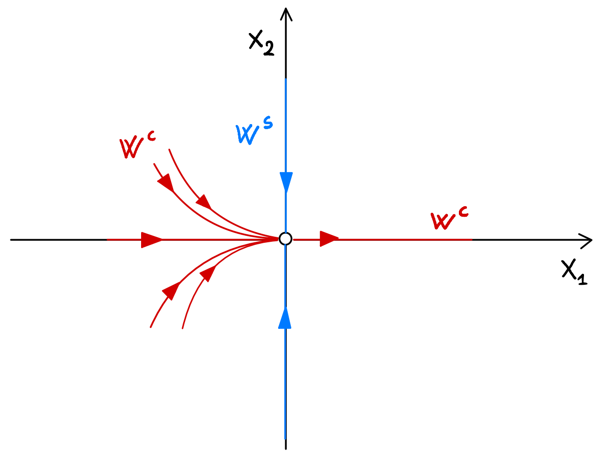



Figure 13: Non-uniqueness of center manifold for the fixed point $\boldsymbol{x}^* = (0,0)$ of the dynamical system (68).

**Stability analysis.** To study stability of the fixed point $\boldsymbol{x}^*$, we need to study the orbits of the nonlinear dynamical system (74) in a neighborhood of $\boldsymbol{q} = \boldsymbol{0}$. To this end, we consider a perturbation of the system (8) depending on one real parameter $\mu$. Such perturbation modifies the eigenvalues of the Jacobian $J_{\boldsymbol{f}}(\boldsymbol{x}^*)$ to

$$\lambda_{1,2} = \alpha(\mu) \pm i\beta(\mu). \tag{75}$$

The real Jordan form of the system (8) after perturbation is

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} H_1(q_1, q_2; \mu) \\ H_2(q_1, q_2; \mu) \end{bmatrix}, \tag{76}$$

where $\alpha(0) = 0$ and $\beta(0) = \omega$.

Note that the perturbation generally changes the location of the fixed point and modifies the vector field in its neighborhood. However, in the normal coordinates $(q_1, q_2)$, the fixed point is always positioned at the origin of the system. It is also worth noting that the flow depends continuously on the perturbation parameter $\mu$, provided the perturbation is sufficiently smooth. Furthermore, a one-parameter perturbation describes the complete local dynamics near $\mu = 0$ if the *transversality condition* $\dot{\alpha}(0) \neq 0$ is satisfied. This guarantees that varying the parameter causes the eigenvalues to cross the imaginary axis.

Let

$$z = q_1 + iq_2. \tag{77}$$

Differentiating with respect to time and using (76) yields

$$\dot{z} = \dot{q}_1 + i\dot{q}_2 = (\alpha + i\beta)z + H_1(q_1, q_2; \mu) + iH_2(q_1, q_2; \mu)$$

Thus, the system (76) reduces to the complex form:

$$\dot{z} = (\alpha + i\beta)z + N(z, \bar{z}; \mu), \tag{78}$$

where the nonlinear term is defined by

$$N(z, \bar{z}; \mu) = H_1(q_1, q_2; \mu) + iH_2(q_1, q_2; \mu),$$

i.e.,

$$\mathrm{Re}(N(z, \bar{z}; \mu)) = H_1(z, \bar{z}; \mu), \qquad \mathrm{Im}(N(z, \bar{z}; \mu)) = H_2(z, \bar{z}; \mu).$$

**Theorem 3.** There exists an analytic (near identity) change of variables $\xi = z + S(z, \bar{z}; \mu)$ with $S = \mathcal{O}(|z|^2)$, such that the system (78) takes the form

$$\dot{\xi} = (\alpha + i\beta)\xi + (\gamma + i\delta)|\xi|^2\xi + \mathcal{O}(|\xi|^4).$$

The proof is given in Appendix B.

Letting $\xi = Re^{i\theta}$, we obtain

$$\dot{R} + iR\dot{\theta} = (\alpha + i\beta)R + \gamma R^3 + i\delta R^3,$$

i.e.,

$$\begin{cases} \dot{R} = \alpha(\mu)R + \gamma(\mu)R^3 \\ \dot{\theta} = \beta(\mu) + \delta(\mu)R^2 \end{cases} \tag{79}$$

This system is written in polar coorinates and it represents the dynamics on the two-dimensional center manifold $W^c$ for small $R$ and small $\mu$.

The equilibria of the first equation in (79) are $R = 0$ and $R^2 = -\alpha(\mu)/\gamma(\mu)$. Since $\alpha(0) = 0$, the sign of $\alpha'(0)$ determines local stability for small $\mu$. If $\alpha'(0) > 0$, then $R = 0$ is unstable for $\mu > 0$ and stable for $\mu < 0$. Conversely, if $\alpha'(0) < 0$, then $R = 0$ is stable for $\mu > 0$ and unstable for $\mu < 0$. Evaluating the system (79) at $\mu = 0$ yields

$$\begin{cases} \dfrac{dR}{dt} = \gamma(0)R^3 \\ \dfrac{d\theta}{dt} = \omega + \delta(0)R^2 \end{cases} \tag{80}$$

Therefore the trajectories nearby the fixed point $\boldsymbol{x}^*$ are either spirals or centers, depending on the parameter $\gamma(0)$. It can be shown (see the proof in Appendix B or the book by Guckenheimer and Holmes, "Nonlinear oscillations, dynamical systems and bifurcations of vector fields", p. 154) that

$$\begin{aligned} \gamma(0) =& \frac{1}{16}\left[\frac{\partial^3 H_1}{\partial q_1^3} + \frac{\partial^3 H_1}{\partial q_1 \partial q_2^2} + \frac{\partial^3 H_2}{\partial q_1^2 \partial q_2} + \frac{\partial^3 H_2}{\partial q_2^3}\right] + \\ & \frac{1}{16\omega}\left[\frac{\partial^2 H_1}{\partial q_1 \partial q_2}\left(\frac{\partial^2 H_1}{\partial q_1^2} + \frac{\partial^2 H_1}{\partial q_2^2}\right) - \frac{\partial^2 H_2}{\partial q_1 \partial q_2}\left(\frac{\partial^2 H_2}{\partial q_1^2} + \frac{\partial^2 H_2}{\partial q_2^2}\right) - \right. \\ & \left. \frac{\partial^2 H_1}{\partial q_1^2}\frac{\partial^2 H_2}{\partial q_1^2} + \frac{\partial^2 H_1}{\partial q_2^2}\frac{\partial^2 H_2}{\partial q_2^2}\right], \end{aligned} \tag{81}$$

where all derivatives of $H_1(q_1, q_2)$ and $H_2(q_1, q_2)$ are evaluated at $(0, 0)$. Hence, if $\gamma(0) < 0$ we get a stable spiral and if $\gamma(0) > 0$ we get an unstable spiral. The case $\gamma(0) = 0$ requires higher order Taylor expansions in Theorem 3.

**Remark:** It is not strictly necessary the take a perturbation of the vector field $H_i(q_1, q_2)$ in (74) to derive the stability condition on the 2D center manifold corresponding to complex conjugate eigenvalues. Indeed, the sequence of steps listed in Appendix B to prove Theorem 3 holds in particular for $\mu = 0$. Adding a parameter helps us understand structural changes in orbits as the parameter is varied.

**Example:** Consider the dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = -x_2 - (x_1^2 + x_2^2) + x_1 x_2 \\ \dfrac{dx_2}{dt} = x_1 - (x_1^2 + x_2^2) - x_1 x_2 \end{cases} \tag{82}$$

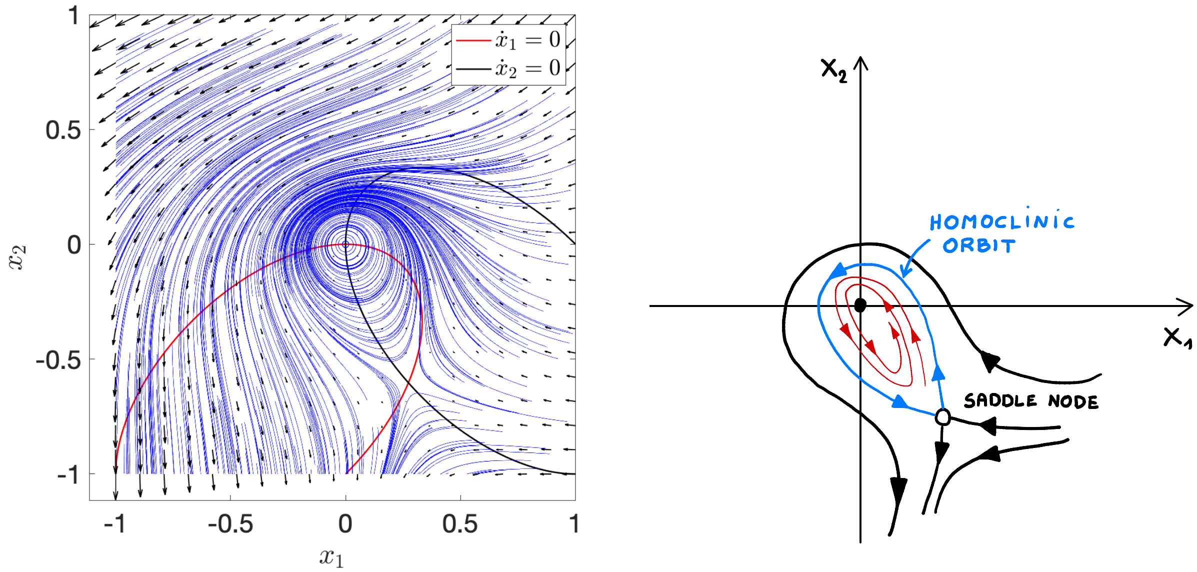



Figure 14: Phase portraint of the system (82). The system has a non-hyperbolic fixed point at $\boldsymbol{x}^* = (0, 0)$, which turns out to be a stable spiral. The stable spiral is enclosed by a *homoclinic trajectory*, i.e., an trajectory that connect the unstable manifold and the stable manifold of the saddle node that is located nearby.

The system has a fixed point at $\boldsymbol{x}^* = (0, 0)$. The Jacobian of (82) at $(0, 0)$ is

$$\boldsymbol{J_f}(\boldsymbol{x}) = \begin{bmatrix} -2x_1 + x_2 & -1 - 2x_2 + x_1 \\ 1 - 2x_1 - x_2 & -2x_2 - x_1 \end{bmatrix} \qquad \Rightarrow \qquad \boldsymbol{J_f}(0, 0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{83}$$

The eigenvalues of $\boldsymbol{J_f}(0, 0)$ are

$$\lambda_{1,2} = \pm i. \tag{84}$$

Hence, $\boldsymbol{x}^* = (0, 0)$ is a *non-hyperbolic* fixed point with an associated two-dimensional center manifold. To study the dynamics nearby $\boldsymbol{x}^* = (0, 0)$ we use the normal form (80) and calculate the coefficient (81) for

$$H_1(x_1, x_2) = -(x_1^2 + x_2^2) + x_1 x_2 \qquad H_2(x_1, x_2) = -(x_1^2 + x_2^2) - x_1 x_2 \tag{85}$$

Note that in this case $\omega$ is equal to one (compare (82) and (74)) and the third derivatives of $(H_1, H_2)$ are both equal to zero. Moreover,

$$\frac{\partial^2 H_1}{\partial x_1 x_2} = 1, \qquad \frac{\partial^2 H_2}{\partial x_1 x_2} = -1, \qquad \frac{\partial^2 H_i}{\partial x_j^2} = -2, \qquad (i, j = 1, 2). \tag{86}$$

Substituting these derivatives in (81) we yields

$$\begin{aligned} \gamma(0) =& \frac{1}{16}\left[1 \times (-2 - 2) - (-1) \times (-2 - 2) - (-2) \times (-2) + (-2) \times (-2)\right] \\ =& \frac{1}{16}\left[-4 - 4 - 4 + 4\right] \\ =& -\frac{1}{2} \end{aligned} \tag{87}$$

Hence, we conclude that the non-hyperbolic fixed point $(0, 0)$ is a *stable spiral.* The phase portrait is for this system is shown in Figure 14. Note that the stable spiral is enclosed by a *homoclinic orbit*, i.e., a trajectory that connects the unstable and unstable manifolds of a nearby saddle point.

### Normal form of $n$-dimensional nonlinear dynamical systems at fixed points

The center manifold Theorem 2 allows us to write any dynamical system in a neighborhood of an equilibrium point in a "normal form". Such normal form differs from a standard linearization in that the dynamics on the subspace $V^c$ is nonlinear. To obtain such normal form let us start from the nonlinear system (8), which represents (1) at the fixed point $\boldsymbol{x}^*$. We group the eigenvalues of the Jacobian matrix $\boldsymbol{J_f}(\boldsymbol{x}^*)$ as in Figure 2, and denote by

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{C} & & \\ & \boldsymbol{S} & \\ & & \boldsymbol{U} \end{bmatrix} \tag{88}$$

the real Jordan form of the Jacobian matrix $\boldsymbol{J_f}(\boldsymbol{x}^*)$. Here $\boldsymbol{C}$ denotes the real Jordan block corresponding to the center subspace, $\boldsymbol{S}$ the real Jordan block corresponding the stable subspace, and $\boldsymbol{U}$ the real Jordan block corresponding to the unstable subspace. The (real) projection matrix $\boldsymbol{P}$ is

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_c & \boldsymbol{P}_s & \boldsymbol{P}_u \end{bmatrix} \tag{89}$$

where $\boldsymbol{P}_c$, $\boldsymbol{P}_s$ and $\boldsymbol{P}_u$ are projection matrices onto the subspaces $V^c$, $V^s$ and $V^u$. Such projection matrices are made of generalized eigenvectors (columnwise) spanning each of the subspaces $V^c$, $V^s$ and $V^u$. The Jordan factorization of $\boldsymbol{J_f}(\boldsymbol{x}^*)$ takes the form

$$\boldsymbol{J_f}(\boldsymbol{x}^*) = \boldsymbol{P}\boldsymbol{K}\boldsymbol{P}^{-1}. \tag{90}$$

Next, define a new set of variables

$$\boldsymbol{q} = \boldsymbol{P}^{-1}\boldsymbol{\eta}. \tag{91}$$

A substitution of (90) and (91) into (8) yields

$$\frac{d\boldsymbol{q}}{dt} = \boldsymbol{K}\boldsymbol{q} + \boldsymbol{P}^{-1}\boldsymbol{g}(\boldsymbol{P}\boldsymbol{q}). \tag{92}$$

Upon definition of

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{s} \\ \boldsymbol{u} \end{bmatrix} \tag{93}$$

this system can be split as

$$\begin{cases} \dfrac{d\boldsymbol{c}}{dt} = & \boldsymbol{C}\boldsymbol{c} + \boldsymbol{f}_c(\boldsymbol{c},\boldsymbol{s},\boldsymbol{u}) \qquad \text{dynamics in } V^c \text{ } (\boldsymbol{C} \text{ has eigenvalues with zero real part}) \\ \dfrac{d\boldsymbol{s}}{dt} = & \boldsymbol{S}\boldsymbol{s} + \boldsymbol{f}_s(\boldsymbol{c},\boldsymbol{s},\boldsymbol{u}) \qquad \text{dynamics in } V^s \text{ } (\boldsymbol{S} \text{ has eigenvalues with negative real part}) \\ \dfrac{d\boldsymbol{u}}{dt} = & \boldsymbol{U}\boldsymbol{u} + \boldsymbol{f}_u(\boldsymbol{c},\boldsymbol{s},\boldsymbol{u}) \qquad \text{dynamics in } V^u \text{ } (\boldsymbol{U} \text{ has eigenvalues with positive real part}) \end{cases} \tag{94}$$

If $\|\boldsymbol{q}\|$ is very small then the nonlinear terms $\boldsymbol{f}_s$ and $\boldsymbol{f}_u$ are negligible with respect to $\boldsymbol{B}\boldsymbol{s}$ and $\boldsymbol{C}\boldsymbol{u}$, respectively. This leaves us with the system

$$\begin{cases} \dfrac{d\boldsymbol{c}}{dt} = & \boldsymbol{C}\boldsymbol{c} + \boldsymbol{f}_c(\boldsymbol{c},\boldsymbol{s},\boldsymbol{u}) \\ \dfrac{d\boldsymbol{s}}{dt} = & \boldsymbol{S}\boldsymbol{s} \\ \dfrac{d\boldsymbol{u}}{dt} = & \boldsymbol{U}\boldsymbol{u} \end{cases} \tag{95}$$

By using the center manifold theorem we can express the dynamics on $W^c$ as a vector map

$$W^c = \{(\boldsymbol{c},\boldsymbol{s},\boldsymbol{u}) \in \mathbb{R}^n : \boldsymbol{s} = \boldsymbol{h}_s(\boldsymbol{c}) \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{h}_u(\boldsymbol{c})\} \tag{96}$$

subject to the conditions

$$\begin{aligned} \boldsymbol{h}_s(\boldsymbol{0}) &= \boldsymbol{0}, \qquad \boldsymbol{h}_u(\boldsymbol{0}) = \boldsymbol{0}, \qquad (W^c \text{ passes through } \boldsymbol{\eta} = \boldsymbol{0}), \\ \nabla \boldsymbol{h}_s(\boldsymbol{0}) &= \boldsymbol{0}, \qquad \nabla \boldsymbol{h}_u(\boldsymbol{0}) = \boldsymbol{0}, \qquad (W^c \text{ is tangent to } V^s \text{ at } \boldsymbol{\eta} = \boldsymbol{0}). \end{aligned} \tag{97}$$

With the center manifold (96) available, we can decouple the system (95) as

$$\begin{cases} \dfrac{d\boldsymbol{c}}{dt} = \boldsymbol{C}\boldsymbol{c} + \boldsymbol{f}_c(\boldsymbol{c}, \boldsymbol{h}_s(\boldsymbol{c}), \boldsymbol{h}_u(\boldsymbol{c})) \\ \dfrac{d\boldsymbol{s}}{dt} = \boldsymbol{S}\boldsymbol{s} \\ \dfrac{d\boldsymbol{u}}{dt} = \boldsymbol{U}\boldsymbol{u} \end{cases} \tag{98}$$

This system of equations represents the generalization of the Hartman-Grobman theorem for non-hyperbolic fixed points. From (98) we see that the dynamics on the stable and unstable subspaces of are trivial in normal coordinates, while the dynamics on the center manifold is essentially nonlinear.

**Example:** Consider a 3D system in a normal form with a 2D center manifold corresponding to a eigenvalue $\lambda_c = 0$ equal to zero with algebraic multiplicity two. Let the other eigenvalue $\lambda_s$ be negative. The system has the form[5]

$$\begin{cases} \dfrac{dc_1}{dt} = c_2 + f_{c_1}(c_1, c_2, s) \\ \dfrac{dc_2}{dt} = f_{c_2}(c_1, c_2, s) \\ \dfrac{ds}{dt} = \lambda_s s \end{cases} \tag{99}$$

To identify the center manifold that is tangent to the 2D center subspace we let

$$s = h_s(c_1, c_2). \tag{100}$$

In normal coordinates we have

$$0 = h_s(0,0) \qquad \text{and} \qquad \frac{\partial h_s(0,0)}{\partial c_1} = \frac{\partial h_s(0,0)}{\partial c_2} = 0. \tag{101}$$

Differentiating (100) with respect to time yields

$$\dot{s} = \frac{\partial h_s}{\partial c_1}\dot{c}_1 + \frac{\partial h_s}{\partial c_2}\dot{c}_2. \tag{102}$$

At this point let the center manifold be represented as a power series

$$h_s(c_1, c_2) = a_0 + a_1 c_1 + a_2 c_2 + a_3 c_1 c_2 + a_4 c_1^2 + a_4 c_2^2 + \cdots \tag{103}$$

The coefficients $a_j$ can be determined by enforcing (101) nd (102) with all time derivatives represented using the right hand side of (99), i.e.,

$$\lambda_3 h_s(c_1, c_2) = \frac{\partial h_s}{\partial c_1} f_{c_1}(c_1, c_2, h_s(c_1, c_2)) + \frac{\partial h_s}{\partial c_2} f_{c_2}(c_1, c_2, h_s(c_1, c_2)). \tag{104}$$

[5] In this case the matrix $\boldsymbol{C}$ is the zero matrix

Once the center manifold $h_s(c_1, c_2)$ is identified, we substitute it into (99) to obtain

$$\begin{cases} \dfrac{d\boldsymbol{c}}{dt} = & \boldsymbol{f}_c(\boldsymbol{c}, h_s(\boldsymbol{c})) \\ \dfrac{ds}{dt} = & \lambda_s s \end{cases} \tag{105}$$

To study stability of the nonlinear equation for $\boldsymbol{c}(t)$ describing the flow on the center manifold we could take perturbations as we did before for the case of complex conjugate eigenvalues, or proceed numerically. Specifically, if the geometric multiplicity of the $\lambda = 0$ is one (i.e., degenerate eigenvalue), it can be shown that after a suitable near-identity change of coordinates, the system can transformed into the form

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = a u_1^2 + b u_1 u_2 + \cdots, \end{cases} \tag{106}$$

where $a, b$ are determined by the system's nonlinearities. To fully explore the local behavior of the system in a neighborhood of the degenerate eigenvalue we need two parameters (in the complex conjugate eigenvalue case the second eigenvalue is fully determined by the first). To fully explore all possible local dynamics we need two parameters$\mu_1$ and $\mu_2$. The system can be written in the Takens–Bogdanov canonical form

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = \mu_1 + \mu_2 u_1 + a u_1^2 + b u_1 u_2 + \cdots, \end{cases} \tag{107}$$

**Lyapunov stability theory**

Stability of hyperbolic and non-hyperbolic fixed points can be also studied using *Lyapunov functions*, without finding the trajectories of (8). A typical Lyapunov Theorem has the form: "if there exists a function $V(\boldsymbol{x})$ that satisfies some conditions on $V(\boldsymbol{X}(t,\boldsymbol{x}_0))$ and $dV(\boldsymbol{X}(t,\boldsymbol{x}_0))/dt$, then the trajectories of the system satisfy some property".

**Theorem 4.** Let $\boldsymbol{f}(\boldsymbol{x})$ be a Lipschitz function defined over a domain $D \subset \mathbb{R}^n$ which contains a fixed point $\boldsymbol{x}^*$, i.e., $\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$. Let $V(\boldsymbol{x})$ be a continuously differentiable function defined over $D$ such that

$$V(\boldsymbol{x}^*) = 0 \qquad \text{and} \qquad V(\boldsymbol{x}) > 0 \qquad \text{for all } \boldsymbol{x} \in D \setminus \{\boldsymbol{x}^*\}, \tag{108}$$

and

$$\frac{dV(\boldsymbol{X}(t,\boldsymbol{x}_0))}{dt} \leq 0 \qquad \forall \boldsymbol{x}_0 \in D. \tag{109}$$

Then, $\boldsymbol{x}^*$ is a stable equilibrium point of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. If

$$\frac{dV(\boldsymbol{X}(t,\boldsymbol{x}_0)))}{dx} < 0 \qquad \forall \boldsymbol{x}_0 \in D \setminus \{\boldsymbol{x}^*\}. \tag{110}$$

then $\boldsymbol{x}^*$ is asymptotically stable[6]. Finally if $D = \mathbb{R}^n$ and (110) holds then $\boldsymbol{x}^*$ is globally asymptotically stable.

The Lyapunov function $\boldsymbol{V}(\boldsymbol{x})$ depends on the vector field (and the flow) generated by a dynamical system. In fact,

$$\frac{dV(\boldsymbol{X}(t,\boldsymbol{x}_0))}{dt} = \nabla \boldsymbol{V}(\boldsymbol{X}(t,x_0)) \cdot \boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)). \tag{112}$$

**Remark**: The conditions in Theorem 4 are only *sufficient*. Failure of a particular Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using that particular Lyapunov function candidate. Whether the equilibrium point is stable (asymptotically stable) or not can be determined only by further investigation.

The following theorem shows that if we can find a Lyapunov function in a domain $D$ about the fixed point $\boldsymbol{x}^*$ whose derivative along the trajectories satisfies (109), and if we can establish that no trajectory can stay identically at points where $dV(\boldsymbol{X}(t,\boldsymbol{x}))/dt = 0$, except at $\boldsymbol{x}^*$, then $\boldsymbol{x}^*$ is *asymptotically stable*. This idea follows from La Salle's invariance principle, which we discuss below.

**Theorem 5** (La Salle)**.** Let $\boldsymbol{f}(\boldsymbol{x})$ be a Lipschitz function defined over a domain $D \subset \mathbb{R}^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to the flow generated by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Let $V(\boldsymbol{x})$ be a continuously differentiable function defined over $D$ such that $dV(\boldsymbol{X}(t,\boldsymbol{x}_0))/dt \leq 0$ for all $\boldsymbol{x}_0 \in \Omega$. Let $\boldsymbol{x}_0 \in E$ be the set of all points in $\Omega$ where $dV(\boldsymbol{X}(t,\boldsymbol{x}_0))/dt = 0$, and $M$ be the largest invariant set in $E$. Then every trajectory $\boldsymbol{X}(t,\boldsymbol{x}_0)$ starting $\boldsymbol{x}_0 \in \Omega$ approaches $M$ as $t \to \infty$.

Unlike Lyapunov's theorem 4, Theorem 5 does not require the function $V(\boldsymbol{x})$ to be positive definite. Note also that the construction of the set $\Omega$ does not have to be tied in with the construction of the function $V(\boldsymbol{x})$.

**Converse Lyapunov Theorems.** Theorem 4 guarantees the asymptotic stability of the fixed point $\boldsymbol{x}^*$ under the existence of a Lyapunov function $V(\boldsymbol{x})$ satisfying certain conditions. However, such a function

[6]Note that the conditions (109) and (110) can be equivalently formulated in an integral form as

$$V(\boldsymbol{X}(t,\boldsymbol{x}_0)) \leq V(\boldsymbol{X}(s,\boldsymbol{x}_0)) \qquad \text{and} \qquad V(\boldsymbol{X}(t,\boldsymbol{x}_0)) < V(\boldsymbol{X}(s,\boldsymbol{x}_0)) \quad \text{for all} \quad t > s, \tag{111}$$

respectively. In other words, the Lyapunov function is non-increasing (or monotonically decreasing) along trajectories of the system is typically not known in advance, and no general method exists for systematically constructing it. This naturally raises the question: does such a function at least exist? The answer is provided by *converse Lyapunov theorems*.

For instance, a converse theorem for asymptotic stability asserts that if $\boldsymbol{x}^*$ is asymptotically stable, then there exists a Lyapunov function satisfying the conditions of Theorem 4. Most converse theorems are proved by explicitly constructing auxiliary functions that meet the required conditions. Unfortunately, these constructions almost always rely on prior knowledge of the solution to the differential equation, limiting their practical use in finding such functions. Nevertheless, converse Lyapunov theorems remain valuable for drawing conceptual insights into the behavior of dynamical systems. One example is presented below.

**Theorem 6.** Let $\boldsymbol{x}^*$ be an asymptotically stable equilibrium point for the $n$-dimensional system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{f}$ is Lipschitz on $D \subset \mathbb{R}^n$ and $\boldsymbol{x}^* \in D$. Let $\Omega \subset D$ be the basin of attraction of $\boldsymbol{x}^*$. Then, there is a smooth, positive function $V(\boldsymbol{x})$ such that $V(\boldsymbol{x}^*) = 0$ and a continuous, positive function $W(\boldsymbol{x})$ such that $W(\boldsymbol{x}^*) = 0$, both defined for all $\boldsymbol{x} \in \Omega$, that satisfy

$$V(\boldsymbol{x}) \to \infty \qquad \text{as} \qquad \boldsymbol{x} \to \partial\Omega \tag{113}$$

and

$$\nabla V(\boldsymbol{X}(t, \boldsymbol{x}_0)) \cdot \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) \leq -W(\boldsymbol{X}(t, \boldsymbol{x}_0)) \qquad \forall \boldsymbol{x}_0 \in \Omega \tag{114}$$

Moreover for each $V(\boldsymbol{x}) \leq c$ defines a compact subset of $\Omega$ for each $c > 0$.

**Basin of attraction of fixed points.** Quite often, it is not sufficient to determine that a given system has an asymptotically stable equilibrium point. Rather, it is important to find the region of attraction of that point (see, e.g., Figure 5), or at least an estimate of it.

**Theorem 7.** The region of attraction of an asymptotically stable equilibrium point is an open, connected, invariant set, and its boundary is formed by trajectories.

This suggests that one way to determine the region of attraction is to characterize the trajectories that lie on its boundary. This process can be quite difficult for high-dimensional systems, but can be easily done for two-dimensional systems by examining phase portraits in the phase plane (see the shaded area in Figure 5).

## Appendix A: Real Jordan form of a 2D matrix with imaginary eigenvalues

In this Appendix we briefly describe the procedure to compute the real Jordan form of a $2 \times 2$ matrix with complex conjugate eigenvalues. The generalization to $n \times n$ matrices with real and complex conjugate eigenvalues is straightforward and can be built based the technique discussed hereafter and in the Appendix A of the course note 4. Let us illustrate how to compute the real Jordan form of a $2 \times 2$ matrix using a simple example. To this end, consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}. \tag{115}$$

The eigenvalues of $\boldsymbol{A}$ are

$$\lambda_{1,2} = \pm\sqrt{3}i, \tag{116}$$

while the eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1+\sqrt{3}i \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1-\sqrt{3}i \end{bmatrix}. \tag{117}$$

Denote by $\overline{\lambda}_i, \overline{\boldsymbol{v}}_i$ the complex conjugates of the eigenvalues and eigenvectors. Clearly, for $i = 1, 2$

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \qquad \Rightarrow \qquad \overline{\boldsymbol{A}\boldsymbol{v}_i} = \overline{\lambda_i \boldsymbol{v}_i} \qquad \Rightarrow \qquad \boldsymbol{A}\overline{\boldsymbol{v}_i} = \overline{\lambda}_i \overline{\boldsymbol{v}_i}, \tag{118}$$

i.e., if $\boldsymbol{v}_i$ is an eigenvector corresponding to $\lambda_i$ then $\overline{\boldsymbol{v}}_i$ is an eigenvector corresponding to $\overline{\lambda}_i$. So, in practice, we just need to compute one eigenvector of $\boldsymbol{A}$, since the other one is going to be the complex conjugate of such vector. To compute the *real Jordan form*, we simply replace the complex eigenvectors (117) with the real and imaginary component of one vector[7], i.e., we consider the real basis

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{bmatrix} \tag{120}$$

We have

$$\boldsymbol{A}\boldsymbol{P} = \underbrace{\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{bmatrix}}_{\boldsymbol{P}} = \begin{bmatrix} 0 & 2\sqrt{3} \\ -3 & -\sqrt{3} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & \sqrt{3} \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{bmatrix}}_{\boldsymbol{J}} \tag{121}$$

Hence the *real Jordan form*[8] is the skew-symmetric matrix

$$\boldsymbol{J} = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{bmatrix} \tag{124}$$

and the similarity transformation (120) has real entries. Of course, we are also allowed to consider the transformation

$$\boldsymbol{P} = \begin{bmatrix} -2 & 0 \\ 1 & -\sqrt{3} \end{bmatrix}, \tag{125}$$

[7] Note that the real component of both vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in (117) is

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{while the imaginary component is} \quad \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}. \tag{119}$$

[8] On the other hand, the *complex Jordan form* is obtained by the methods we studies in the course note 4. In fact the matrix $\boldsymbol{A}$ is diagonalizable Hence, we have

$$\boldsymbol{J} = \begin{bmatrix} \sqrt{3}i & 0 \\ 0 & -\sqrt{3}i \end{bmatrix} \tag{122}$$

and the (complex) similarity transformation

$$\boldsymbol{P} = \begin{bmatrix} -2 & -2 \\ 1-\sqrt{3}i & 1+\sqrt{3}i \end{bmatrix}. \tag{123}$$

which yields the real Jordan form

$$\boldsymbol{J} = \begin{bmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}. \tag{126}$$

If a $2 \times 2$ matrix $\boldsymbol{A}$ has complex conjugate eigenvalues of the form

$$\lambda_{1,2} = \mu \pm i\omega \tag{127}$$

then the real Jordan form of $\boldsymbol{A}$ is

$$\boldsymbol{J} = \begin{bmatrix} \mu & \pm\omega \\ \mp\omega & \mu \end{bmatrix}. \tag{128}$$

## Appendix B: Proof of Theorem 3

We consider the complex differential equation (78), hereafter rewritten for convenience

$$\dot{z} = (\alpha + i\beta)z + N(z, \bar{z}; \mu),$$

where $z \in \mathbb{C}$, and the nonlinear term $N$ is analytic and satisfies $N(z, \bar{z}; \mu) = \mathcal{O}(|z|^2)$. We expand $N$ up to cubic order

$$N(z, \bar{z}; \mu) = \frac{1}{2}n_1 z^2 + n_2 z\bar{z} + \frac{1}{2}n_3 \bar{z}^2 + \frac{1}{6}n_4 z^3 + \frac{1}{2}n_5 z^2\bar{z} + \frac{1}{2}n_6 z\bar{z}^2 + \frac{1}{6}n_7 \bar{z}^3 + \mathcal{O}(|z|^4),$$

where all coefficients $n_j = n_j(\mu) \in \mathbb{C}$ depend smoothly on the parameter $\mu$. We now apply a quadratic near-identity transformation

$$\eta = z + S(z, \bar{z}; \mu), \quad S(z, \bar{z}; \mu) = \frac{1}{2}a_1 z^2 + a_2 z\bar{z} + \frac{1}{2}a_3 \bar{z}^2,$$

with unknown coefficients $a_1, a_2, a_3 \in \mathbb{C}$. Differentiating and substituting for $\dot{z}$, we have:

$$\begin{aligned}
\dot{\eta} &= \dot{z} + D_z S(z, \bar{z}) \cdot \dot{z} + D_{\bar{z}} S(z, \bar{z}) \cdot \dot{\bar{z}} + \mathcal{O}(|z|^3) \\
&= (\alpha + i\beta)z + N(z, \bar{z}; \mu) + (a_1 z + a_2 \bar{z})\dot{z} + (a_2 z + a_3 \bar{z})\dot{\bar{z}} + \mathcal{O}(|z|^3)
\end{aligned}$$

Now substitute the inverse transformation

$$z = \eta - \frac{1}{2}a_1 \eta^2 - a_2 \eta\bar{\eta} - \frac{1}{2}a_3 \bar{\eta}^2 + \mathcal{O}(|\eta|^3)$$

and use it to rewrite $\dot{\eta}$ in terms of $\eta, \bar{\eta}$. The result is

$$\begin{aligned}
\dot{\eta} &= (\alpha + i\beta)\eta \\
&\quad + \frac{1}{2}\big(n_1 + (\alpha + i\beta)a_1\big)\eta^2 + \big(n_2 + (\alpha + i\beta)a_2\big)\eta\bar{\eta} + \frac{1}{2}\big(n_3 + (\alpha + 3i\beta)a_3\big)\bar{\eta}^2 \\
&\quad + \frac{1}{6}n_4 \eta^3 + \frac{1}{2}n_5 \eta^2\bar{\eta} + \frac{1}{2}n_6 \eta\bar{\eta}^2 + \frac{1}{6}n_7 \bar{\eta}^3 + \mathcal{O}(|\eta|^4)
\end{aligned}$$

To eliminate the terms $\eta^2$ and $\bar{\eta}^2$ we set

$$a_1 = -\frac{n_1}{\alpha + i\beta}, \quad a_3 = -\frac{n_3}{\alpha + 3i\beta}.$$

This gives the "intermediate" normal form

$$\dot{\eta} = (\alpha + i\beta)\eta + \hat{n}_2 \eta\bar{\eta} + \frac{1}{6}n_4 \eta^3 + \frac{1}{2}n_5 \eta^2\bar{\eta} + \frac{1}{2}n_6 \eta\bar{\eta}^2 + \frac{1}{6}n_7 \bar{\eta}^3 + \mathcal{O}(|\eta|^4),$$

where $\hat{n}_2 = n_2 + (\alpha + i\beta)a_2$. We now apply a second near-identity transformation

$$\xi = \eta + R(\eta, \bar{\eta}; \mu), \quad R = \frac{1}{3}r_1\eta^3 + r_2\eta^2\bar{\eta} + r_3\eta\bar{\eta}^2 + \frac{1}{3}r_4\bar{\eta}^3.$$

Differentiating again and substituting as before yields

$$\begin{aligned}\dot{\xi} &= (\alpha + i\beta)\xi \\ &\quad + \frac{1}{3}(n_4 + 2r_1(\alpha + i\beta))\xi^3 + (\hat{n}_2 + 2r_2\alpha)\xi^2\bar{\xi} \\ &\quad + (n_6 + 2r_3(\alpha + i\beta))\xi\bar{\xi}^2 + \frac{1}{3}(n_7 + 2r_4(\alpha + 4i\beta))\bar{\xi}^3 + \mathcal{O}(|\xi|^4)\end{aligned}$$

Now choose

$$r_1 = -\frac{1}{2(\alpha + i\beta)}n_4, \quad r_3 = -\frac{1}{2(\alpha + i\beta)}n_6, \quad r_4 = -\frac{1}{2(\alpha + 4i\beta)}n_7, \tag{129}$$

and leave $r_2 = 0$. This yields

$$\dot{\xi} = (\alpha + i\beta)\xi + (\gamma + i\delta)|\xi|^2\xi + \mathcal{O}(|\xi|^4),$$

where $\gamma + i\delta = \hat{n}_2 = n_2 + (\alpha + i\beta)a_2$. This completes the proof.