Liouville theorem

In this course note, we study how the volume of a compact region $D_0 \subset \mathbb{R}^n$ evolves over time as its points $\boldsymbol{x}_0 \in D_0$ are advected by the flow generated by the n-dimensional nonlinear dynamical system

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}), \\ \boldsymbol{x}(0) = \boldsymbol{x}_0. \end{cases} \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is at least continuously differentiable in $\boldsymbol{x}$. To this end, recall that the volume of a region $D(t)$ advected by the flow generated by (1) can be expressed as (see Figure 1)

$$V(t) = \int_{D(t)} 1 d\boldsymbol{x} \tag{2}$$

where $d\boldsymbol{x} = dx_1 \cdots dx_n$. Since the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$ is invertible, we can transform the coordinates back to $\boldsymbol{x}_0$ and write the integral (2) as

$$V(t) = \int_{D_0} |J(t, \boldsymbol{x}_0)| \, d\boldsymbol{x}_0, \tag{3}$$

where

$$J(t, \boldsymbol{x}_0) = \det\left(\frac{\partial \boldsymbol{X}(t, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0}\right) = \det\left(\begin{bmatrix} \dfrac{\partial X_1(t, \boldsymbol{x}_0)}{\partial x_{01}} & \cdots & \dfrac{\partial X_1(t, \boldsymbol{x}_0)}{\partial x_{0n}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial X_n(t, \boldsymbol{x}_0)}{\partial x_{01}} & \cdots & \dfrac{\partial X_n(t, \boldsymbol{x}_0)}{\partial x_{0n}} \end{bmatrix}\right) \tag{4}$$

is the Jacobian determinant of the coordinate change[1] $\boldsymbol{X}(t, \boldsymbol{x}_0) \leftrightarrow \boldsymbol{x}_0$ at each time t.

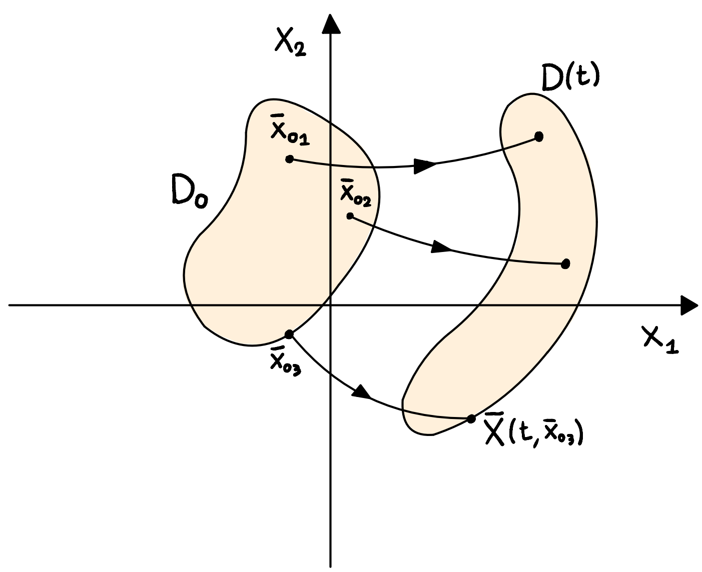

Figure 1: Illustration of how a domain $D_0 \subset \mathbb{R}^2$ is transported over time to a region $D(t) \subset \mathbb{R}^2$ by the flow $\boldsymbol{X}(t, \boldsymbol{x}_0)$ generated by the dynamical system (1). The rate of change of the area (or volume) of $D(t)$, denoted $\frac{dV(t)}{dt}$, is governed by Liouville's Theorem 2.

[1]We know that the flow map $\boldsymbol{X}(t, \boldsymbol{x}_0)$ generated by a smooth dynamical system is invertible at each point where the solution to (1) exists and is unique.

Theorem 1. Let $\boldsymbol{X}(t, \boldsymbol{x}_0)$ be the flow generated by (1). The Jacobian determinant of $\boldsymbol{X}(t, \boldsymbol{x}_0)$ satisfies the linear differential equation

$$\frac{\partial J(t, \boldsymbol{x}_0)}{\partial t} = \nabla \cdot \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) J(t, \boldsymbol{x}_0) \tag{5}$$

Proof. Let us first prove the theorem for two-dimensional dynamical systems. In this case, the determinant (4) can be written explicitly as

$$J(t, \boldsymbol{x}_0) = \det\left(\frac{\partial \boldsymbol{X}(t, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0}\right) = \frac{\partial X_1}{\partial x_{01}} \frac{\partial X_2}{\partial x_{02}} - \frac{\partial X_2}{\partial x_{01}} \frac{\partial X_1}{\partial x_{02}}. \tag{6}$$

Differentiate (6) with respect to time to t to obtain

$$\begin{aligned}\frac{\partial J(t, \boldsymbol{x}_0)}{\partial t} =& \frac{\partial}{\partial x_{10}}\left(\frac{dX_1(t, \boldsymbol{x}_0)}{dt}\right)\frac{\partial X_2(t, \boldsymbol{x}_0)}{\partial x_{20}} + \frac{\partial X_1(t, \boldsymbol{x}_0)}{\partial x_{10}}\frac{\partial}{\partial x_{20}}\left(\frac{dX_2(t, \boldsymbol{x}_0)}{dt}\right) - \\ & \frac{\partial}{\partial x_{20}}\left(\frac{dX_1(t, \boldsymbol{x}_0)}{dt}\right)\frac{\partial X_2(t, \boldsymbol{x}_0)}{\partial x_{10}} - \frac{\partial X_1(t, \boldsymbol{x}_0)}{\partial x_{20}}\frac{\partial}{\partial x_{10}}\left(\frac{dX_2(t, \boldsymbol{x}_0)}{dt}\right).\end{aligned} \tag{7}$$

At this point we recall that

$$\frac{dX_i(t, \boldsymbol{x}_0)}{dt} = f_i\left(X_1(t, \boldsymbol{x}_0), X_2(t, \boldsymbol{x}_0)\right), \qquad i = 1, 2, \tag{8}$$

which implies that

$$\begin{aligned}
\frac{\partial}{\partial x_{10}}\left(\frac{dX_1(t, \boldsymbol{x}_0)}{dt}\right) &= \frac{\partial f_1\left(X_1(t, \boldsymbol{x}_0), X_2(t, \boldsymbol{x}_0)\right)}{\partial x_{10}} = \frac{\partial f_1}{\partial x_1}\frac{\partial X_1}{\partial x_{10}} + \frac{\partial f_1}{\partial x_2}\frac{\partial X_2}{\partial x_{10}}, \\
\frac{\partial}{\partial x_{20}}\left(\frac{dX_1(t, \boldsymbol{x}_0)}{dt}\right) &= \frac{\partial f_1\left(X_1(t, \boldsymbol{x}_0), X_2(t, \boldsymbol{x}_0)\right)}{\partial x_{20}} = \frac{\partial f_1}{\partial x_1}\frac{\partial X_1}{\partial x_{20}} + \frac{\partial f_1}{\partial x_2}\frac{\partial X_2}{\partial x_{20}}, \\
\frac{\partial}{\partial x_{10}}\left(\frac{dX_2(t, \boldsymbol{x}_0)}{dt}\right) &= \frac{\partial f_2\left(X_1(t, \boldsymbol{x}_0), X_2(t, \boldsymbol{x}_0)\right)}{\partial x_{10}} = \frac{\partial f_2}{\partial x_1}\frac{\partial X_1}{\partial x_{10}} + \frac{\partial f_2}{\partial x_2}\frac{\partial X_2}{\partial x_{10}}, \\
\frac{\partial}{\partial x_{20}}\left(\frac{dX_2(t, \boldsymbol{x}_0)}{dt}\right) &= \frac{\partial f_2\left(X_1(t, \boldsymbol{x}_0), X_2(t, \boldsymbol{x}_0)\right)}{\partial x_{20}} = \frac{\partial f_2}{\partial x_1}\frac{\partial X_1}{\partial x_{20}} + \frac{\partial f_2}{\partial x_2}\frac{\partial X_2}{\partial x_{20}}.
\end{aligned}$$

A substitution of these expressions back into (7) yields

$$\begin{aligned}
\frac{\partial J(t, \boldsymbol{x}_0)}{\partial t} =& \frac{\partial f_1}{\partial x_1}\frac{\partial X_1}{\partial x_{10}}\frac{\partial X_2}{\partial x_{20}} + \frac{\partial f_1}{\partial x_2}\frac{\partial X_2}{\partial x_{10}}\frac{\partial X_2}{\partial x_{20}} + \frac{\partial f_2}{\partial x_1}\frac{\partial X_1}{\partial x_{20}}\frac{\partial X_1}{\partial x_{10}} + \frac{\partial f_2}{\partial x_2}\frac{\partial X_2}{\partial x_{20}}\frac{\partial X_1}{\partial x_{10}} - \\
& \frac{\partial f_1}{\partial x_1}\frac{\partial X_1}{\partial x_{20}}\frac{\partial X_2}{\partial x_{10}} - \frac{\partial f_1}{\partial x_2}\frac{\partial X_2}{\partial x_{20}}\frac{\partial X_2}{\partial x_{10}} - \frac{\partial f_2}{\partial x_1}\frac{\partial X_1}{\partial x_{10}}\frac{\partial X_1}{\partial x_{20}} - \frac{\partial f_2}{\partial x_2}\frac{\partial X_2}{\partial x_{10}}\frac{\partial X_1}{\partial x_{20}}, \\
=& \underbrace{\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)}_{\nabla \cdot \boldsymbol{f}} J(t, \boldsymbol{x}_0),
\end{aligned}$$

which proves the theorem in two dimensions. To prove the theorem in n dimensions, let

$$\boldsymbol{F}(t, \boldsymbol{x}_0) = \frac{\partial \boldsymbol{X}(t, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0},$$

so that $J(t, \boldsymbol{x}_0) = \det(\boldsymbol{F}(t, \boldsymbol{x}_0))$. The matrix $\boldsymbol{F}$ satisfies the equation

$$\frac{d\boldsymbol{F}(t, \boldsymbol{x}_0)}{dt} = \boldsymbol{J}_{\boldsymbol{f}}(\boldsymbol{X}(t, \boldsymbol{x}_0))\boldsymbol{F}(t, \boldsymbol{x}_0),$$

where $\boldsymbol{J_f}(\boldsymbol{X}(t,\boldsymbol{x}_0))$ is the Jacobian matrix of $\boldsymbol{f}$ evaluated along the flow $\boldsymbol{X}(t,\boldsymbol{x}_0)$. Using Jacobi's formula for the derivative of a determinant:

$$\begin{aligned}\frac{d}{dt}J(t,\boldsymbol{x}_0) =& J(t,\boldsymbol{x}_0)\,\mathrm{trace}\left(\boldsymbol{F}(t,\boldsymbol{x}_0)^{-1}\frac{d\boldsymbol{F}(t,\boldsymbol{x}_0)}{dt}\right)\\ =& J(t,\boldsymbol{x}_0)\,\mathrm{trace}\left(\boldsymbol{F}(t,\boldsymbol{x}_0)^{-1}\boldsymbol{J_f}(\boldsymbol{X}(t,\boldsymbol{x}_0))\boldsymbol{F}(t,\boldsymbol{x}_0)\right). \end{aligned} \tag{9}$$

Since similarity transformations do not change the matrix trace[2] we have

$$\mathrm{trace}\left(\boldsymbol{F}(t,\boldsymbol{x}_0)^{-1}\boldsymbol{J_f}(\boldsymbol{X}(t,\boldsymbol{x}_0))\boldsymbol{F}(t,\boldsymbol{x}_0)\right) = \mathrm{trace}\left(\boldsymbol{J_f}(\boldsymbol{X}(t,\boldsymbol{x}_0))\right) = \nabla\cdot\boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)).$$

Therefore,

$$\frac{dJ(t,\boldsymbol{x}_0)}{dt} = J(t,\boldsymbol{x}_0)\nabla\cdot\boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)),$$

which completes the proof.

□

Note that at $t=0$ we have $\boldsymbol{X}(0,\boldsymbol{x}_0)=\boldsymbol{x}_0$ and therefore

$$\det\left(\frac{\partial\boldsymbol{X}(0,\boldsymbol{x}_0)}{\partial\boldsymbol{x}_0}\right) = \det(\boldsymbol{I}) = 1. \tag{11}$$

With this initial condition we integrate the (separable) ODE (5) to obtain

$$J(t,\boldsymbol{x}_0) = \exp\left[\int_0^t \nabla\cdot\boldsymbol{f}(\boldsymbol{X}(\tau,\boldsymbol{x}_0))d\tau\right] \tag{12}$$

We now have all element to prove the following theorem.

Theorem 2 (Liouville's theorem)**.** The volume $V(t)$ of a compact region $D(t)\subset\mathbb{R}^n$ advected by the flow $\boldsymbol{X}(t,\boldsymbol{x}_0)$ generated by the smooth dynamical system (1) satisfies

$$\frac{dV(t)}{dt} = \int_{D(t)}\nabla\cdot\boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x} \tag{13}$$

Proof. The volume of $D(t)$ can be expressed as (see equation (3))

$$V(t) = \int_{D_0}|J(t,\boldsymbol{x}_0)|\,d\boldsymbol{x}_0. \tag{14}$$

Thanks to (12), we have that $J\geq 0$. Therefore we can disregard the absolute value in $|J(t,\boldsymbol{x}_0)|$. Differentiating with respect to time and using (5) yields

$$\begin{aligned}\frac{dV(t)}{dt} =& \frac{d}{dt}\int_{D_0}J(t,\boldsymbol{x}_0)d\boldsymbol{x}_0\\ =& \int_{D_0}\frac{\partial J(t,\boldsymbol{x}_0)}{\partial t}d\boldsymbol{x}_0\\ =& \int_{D_0}\nabla\cdot\boldsymbol{f}\left(\boldsymbol{X}(t,\boldsymbol{x}_0)\right)J(t,\boldsymbol{x}_0)d\boldsymbol{x}_0\\ =& \int_{D(t)}\nabla\cdot\boldsymbol{f}\left(\boldsymbol{x}\right)d\boldsymbol{x}.\end{aligned} \tag{15}$$

[2]Recall that for any two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size we have

$$\mathrm{trace}(\boldsymbol{AB}) = \mathrm{trace}(\boldsymbol{BA}) \tag{10}$$

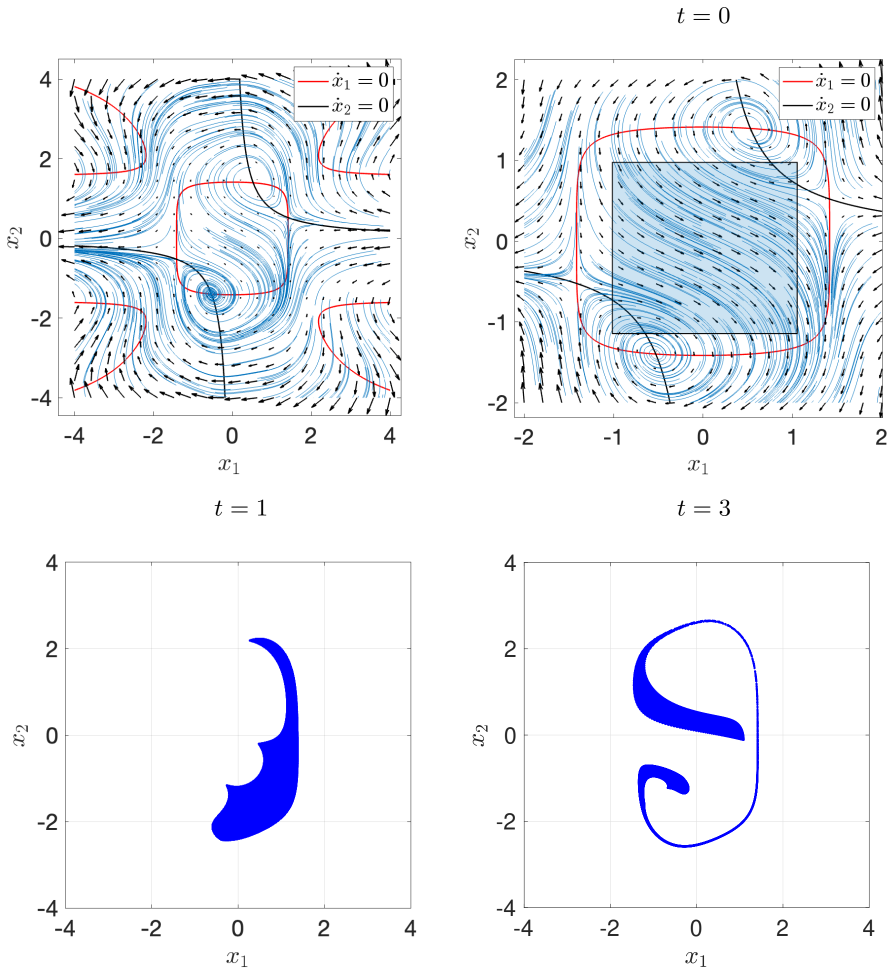

Figure 2: Phase portrait of the system (17) together with two temporal snapshots how $D_0 = [-1, 1] \times [-1, 1]$ looks like at $t = 1$ and $t = 3$ under the flow generated by the nonlinear system.

□

Hence, if the divergence of the vector field $\boldsymbol{f}(\boldsymbol{x})$ is negative for all $\boldsymbol{x} \in \mathbb{R}^n$ then the volume of any phase space domain D_0 *shrinks to zero* (volume contracting) as time evolves. Similarity, if the divergence is zero, i.e., $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = 0$ then the system is *volume-preserving*[3].

A substitution of (12) into (14) yields

$$V(t) = \int_{D_0} \exp\left[\int_0^t \nabla \cdot \boldsymbol{f}(\boldsymbol{X}(\tau, \boldsymbol{x}_0)) d\tau\right] d\boldsymbol{x}_0. \tag{16}$$

This formula allows us to calculate the time evolution of the volume of any domain D_0 in the phase space.

Example: Consider the nonlinear dynamical system

$$\begin{cases} \dot{x}_1 = 2 - x_2^2 - x_1^2 \cos(x_2) \\ \dot{x}_2 = x_1 x_2 - \cos(x_1 x_2) \end{cases} \tag{17}$$

[3]As we will see, Hamiltonian dynamical systems are volume-preserving.

The phase portrait is shown in Figure 2 together with two temporal snapshots representing the position of $D_0 = [2,3] \times [-11]$ at two different times. The divergence of the vector field is

$$\nabla \cdot \boldsymbol{f} = x_1 \left(1 - 2\cos(x_2) + \sin(x_1 x_2)\right) \tag{18}$$

and it can be positive or negative, suggesting that there are regions of the phase plane that are contracting and some other that are expanding.

Lemma 1. The divergence ofthe vector field $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ equals the trace of the matrix $\boldsymbol{A}$. Hence, for linear dynamical system equation (16) reduces to

$$V(t) = V_0 e^{\text{trace}(\boldsymbol{A})t}. \tag{19}$$

Proof. We only need to show that $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = \text{trace}(\boldsymbol{A})$. To this end,

$$\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = \sum_{j=1}^{n} \frac{\partial f_j(\boldsymbol{x})}{\partial x_j} = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial (A_{jk} x_k)}{\partial x_j} = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk} \frac{\partial x_k}{\partial x_j} = \sum_{j=1}^{n} A_{jj} = \text{trace}(\boldsymbol{A}). \tag{20}$$

□

Example: Consider the linear dynamical system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -3x_1 - 2x_2 \end{cases} \tag{21}$$

The eigenvalues of matrix $\boldsymbol{A}$ associated with the system are

$$\lambda_{1,2} = \frac{-3}{2} \pm \frac{i\sqrt{11}}{2}.$$

Therefore the origin is a stable spiral. We aim to determine the time it takes for the unit square $D_0 = [0,1]^2$, which initially has area $V_0 = 1$, to contract to an area of $1/3$ under the flow generated by (21). Using formula (19), and noting that $\text{trace}(\boldsymbol{A}) = -3$, we obtain

$$V(t^*) = \frac{V_0}{3} = \frac{1}{3} = e^{-3t^*} \quad \Rightarrow \quad t^* = \frac{\log(3)}{3}. \tag{22}$$

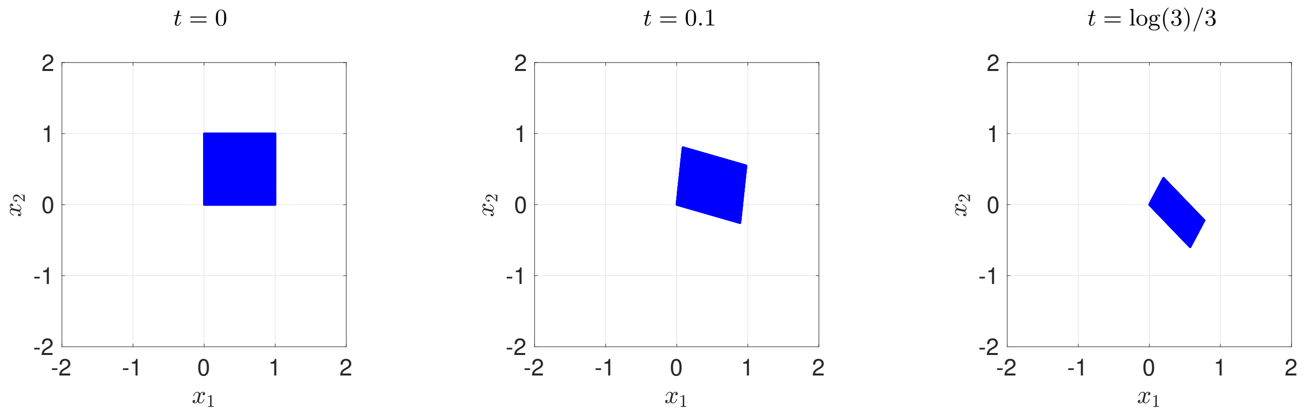

Figure 3: Dynamics of the unit square $D_0 = [0,1]^2$ under the flow generated by (21). At time $t = \log(3)/3$ the area of $D(t)$ is exacly 1/3.

Liouville equation

Let the initial condition of the system (1) be random with probability density function (PDF) $p(0, \boldsymbol{x}_0)$. Our goal is to derive an equation that describes the evolution of the PDF associated with $\boldsymbol{X}(t, \boldsymbol{x}_0)$, the flow generated by the system (1). This flow transforms the random vector $\boldsymbol{x}_0$, distributed according to the initial PDF, into a new random vector $\boldsymbol{X}(t, \boldsymbol{x}_0)$. The regularity of this transformation depends on the smoothness of the vector field $\boldsymbol{f}(\boldsymbol{x})$.

Theorem 3. Let $p(0, \boldsymbol{x}_0) = p_0(x_0)$ be the PDF of the initial condition in (1). The PDF of $\boldsymbol{X}(t, \boldsymbol{x}_0)$, where $\boldsymbol{X}(t, \boldsymbol{x}_0)$ is the flow generated by the ODE (1), satisfies the linear first-order transport PDE

$$\begin{cases} \dfrac{\partial p(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot (\boldsymbol{f}(\boldsymbol{x}) p(\boldsymbol{x}, t)) = 0 \qquad \boldsymbol{x} \in \mathbb{R}^d \qquad t \geq 0 \\ p(\boldsymbol{x}, 0) = p_0(\boldsymbol{x}) \end{cases} \tag{23}$$

Proof. We provide a proof based on the principle of mass conservation. To this end, let $D(t) \subseteq \mathbb{R}^n$ be a domain large enough to contain the support of the probability density function $p(\boldsymbol{x}, t)$. Then,

$$\int_{D(t)} p(\boldsymbol{x}, t)\, d\boldsymbol{x} = 1 \qquad \text{for all } t \geq 0. \tag{24}$$

Let

$$J(t, \boldsymbol{x}_0) = \det \left(\frac{\partial \boldsymbol{X}(t, \boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} \right) \tag{25}$$

be the Jacobian determinant of the flow map. Differentiating both sides of (24) with respect to time gives

$$\begin{aligned} \frac{d}{dt} \int_{D(t)} p(\boldsymbol{x}, t) d\boldsymbol{x} &= \int_{D_0} \frac{\partial}{\partial t} \left(p(\boldsymbol{X}(t, \boldsymbol{x}_0), t) J(t, \boldsymbol{x}_0) \right) d\boldsymbol{x}_0 \\ &= \int_{D_0} \Bigg(\frac{\partial p(\boldsymbol{X}(t, \boldsymbol{x}_0), t)}{\partial t} + \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) \cdot \nabla p(\boldsymbol{X}(t, \boldsymbol{x}_0), t) + \\ &\qquad\qquad p(\boldsymbol{X}(t, \boldsymbol{x}_0)) \nabla \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)) \Bigg) J(t, \boldsymbol{x}_0) d\boldsymbol{x}_0 \\ &= \int_{D(t)} \left[\frac{\partial p(\boldsymbol{x}, t)}{\partial t} + \nabla \left(\boldsymbol{f}(\boldsymbol{x}) p(\boldsymbol{x}, t) \right) \right] d\boldsymbol{x} \\ &= 0. \end{aligned} \tag{26}$$

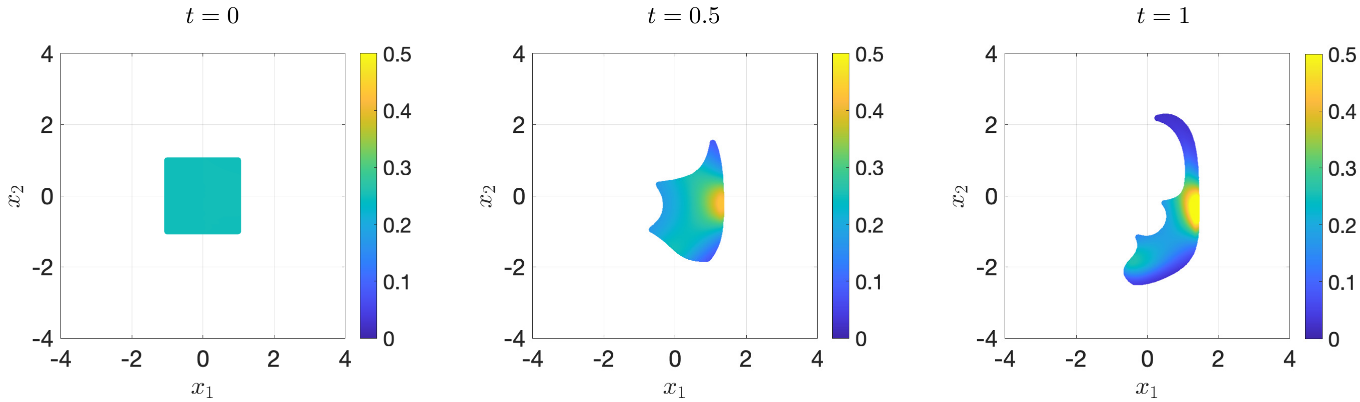

Figure 4: Temporal evolution of a uniform PDF on $D_0 = [-1, 1] \times [-1, 1]$ under the flow generated by the system (17) (see Figure 2 for the phase portrait).

Since $D(t)$ is arbitrary, the integrand must vanish pointwise. Therefore,

$$\frac{\partial p(\boldsymbol{x}, t)}{\partial t} + \nabla \cdot (\boldsymbol{f}(\boldsymbol{x})\, p(\boldsymbol{x}, t)) = 0, \tag{27}$$

which completes the proof.

□

By integrating (23) with the method of characteristics we obtain the following formal solution

$$p(\boldsymbol{x}, t) = p_0(\boldsymbol{X}_0(t, \boldsymbol{x})) \exp\left[-\int_0^t \nabla \cdot \boldsymbol{f}(\boldsymbol{X}(s, \boldsymbol{x}_0)) ds\right]_{\boldsymbol{x}_0 = \boldsymbol{X}_0(t, \boldsymbol{x})} \tag{28}$$

where $\boldsymbol{X}(t, \boldsymbol{x}_0)$ and $\boldsymbol{X}_0(t, \boldsymbol{x})$ denote the forward and inverse flow, respectively. Clearly, if the flow is generated by a divergence-free vector field, i.e. the flow is volume-preserving, then

$$p(\boldsymbol{x}, t) = p_0(\boldsymbol{X}_0(t, \boldsymbol{x})). \tag{29}$$

Example: In Figure 4 we plot the time evolution of a uniform PDF on $D_0 = [-1, 1] \times [-1, 1]$ under the flow generated by the system (17)

Appendix A: Solving the Liouville equation with the method of characteristics

The Liouville equation (23) can be written as

$$\begin{cases} \dfrac{\partial p(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{f}(\boldsymbol{x}) \cdot \nabla p(\boldsymbol{x}, t) = -p(\boldsymbol{x}, t) \nabla \cdot f(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^d \qquad t \geq 0 \\ p(\boldsymbol{x}, 0) = p_0(\boldsymbol{x}) \end{cases} \tag{30}$$

As is well-known, this equation can be transformed into an ODE along the flow generated by the nonlinear dynamical system

$$\frac{d\boldsymbol{X}(t, \boldsymbol{x}_0)}{dt} = \boldsymbol{f}(\boldsymbol{X}(t, \boldsymbol{x}_0)), \qquad \boldsymbol{X}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0. \tag{31}$$

To this end, define

$$p(t) = p(\boldsymbol{X}(t, \boldsymbol{x}_0), t), \tag{32}$$

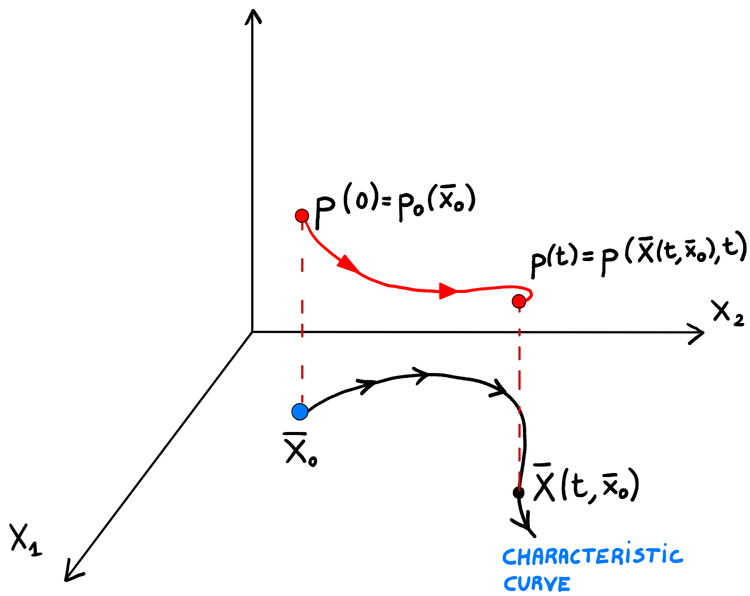

Figure 5: Sketch of the method of characteristics applied to the Liouville equation (30).

that is, the probability density function (PDF) evaluated along the flow generated by (31). Differentiating $p(t)$ with respect to time yields

$$\frac{dp(t)}{dt} = \frac{\partial p(\boldsymbol{X}(t,\boldsymbol{x}_0),t)}{\partial t} + \boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)) \cdot \nabla p(\boldsymbol{X}(t,\boldsymbol{x}_0),t) = -p(t)\nabla \cdot \boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)), \tag{33}$$

which gives

$$\frac{dp}{dt} = -p(t)\nabla \cdot \boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)), \qquad p(0) = p_0(\boldsymbol{x}_0). \tag{34}$$

For each fixed $\boldsymbol{x}_0$ and initial density $p_0(\boldsymbol{x}_0)$, the system

$$\begin{cases} \dfrac{d\boldsymbol{X}(t,\boldsymbol{x}_0)}{dt} = \boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)), \\ \dfrac{dp(t)}{dt} = -p(t)\nabla \cdot \boldsymbol{f}(\boldsymbol{X}(t,\boldsymbol{x}_0)), \\ \boldsymbol{X}(0,\boldsymbol{x}_0) = \boldsymbol{x}_0, \\ p(0) = p_0(\boldsymbol{x}_0), \end{cases} \tag{35}$$

allows us to compute the probability density function $p(\boldsymbol{x},t)$ along a trajectory of the dynamical system (1) (see Figure 5). If we are interested in the solution of (30) at a particular point in space, say $\boldsymbol{x}^*$ (e.g., a point on a spatial grid), and at a specific time t^*, we can proceed as follows:

1. Integrate the characteristic equation (31) backward in time from $t = t^*$ to $t = 0$ with initial condition $\boldsymbol{x}^*$. This yields the point $\boldsymbol{x}_0^*$, as illustrated in Figure 6.

2. With $\boldsymbol{x}_0^*$ known, integrate equation (34) forward in time from $t = 0$ to $t = t^*$.

This procedure allows us to compute the solution of (30) at time $t = t^*$ for all points on a given spatial grid. To do so, we simply apply the above method at each grid point: solve (31) backward in time, followed by solving (34) forward in time.

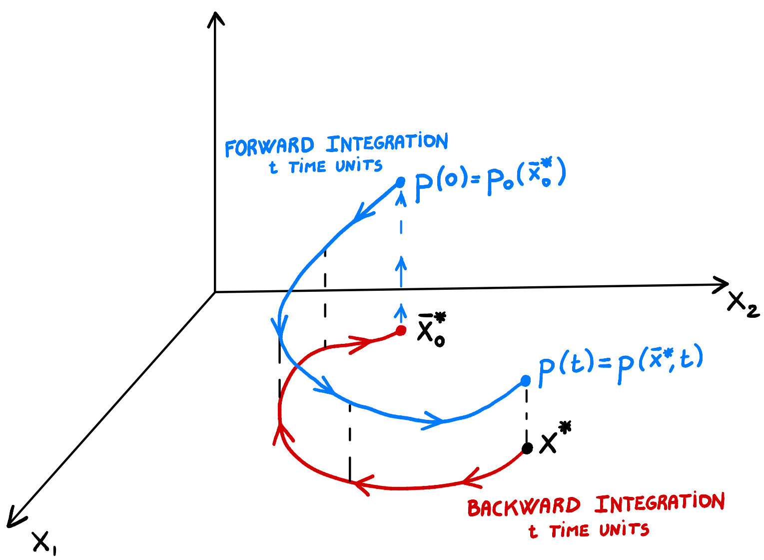

Figure 6: Sketch of the process used to compute the solution of the Liouville equation (30) at a particular point $\boldsymbol{x}^*$ and particular time t^*. Essentially, we can just integrate the characteristic system (31) backward in time from $t = t^*$ and position $\boldsymbol{x}^*$ to $t = 0$. This gives us the point $\boldsymbol{x}_0^*$. Then we integrate (34) forward in time with initial condition $p(0) = p(\boldsymbol{x}_0^*)$ along the same characteristic curve.