

## Conservative and gradient systems

**Conservative systems.** We say the  $n$ -dimensional nonlinear dynamical system

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \quad (1)$$

is *conservative* if there exists a scalar field  $E(\mathbf{x})$ , i.e., a function from  $\mathbb{R}^n$  into  $\mathbb{R}$ , that is constant<sup>1</sup> along each trajectory of (1) and non-constant in any open set  $D \subseteq \mathbb{R}^n$ . This definition implies that trajectories of a conservative system (1) are level sets of  $E(\mathbf{x})$ , i.e.,

$$E(\mathbf{X}(t, \mathbf{x}_0)) = E(\mathbf{x}_0), \quad (2)$$

where  $\mathbf{X}(t, \mathbf{x}_0)$  is the trajectory generated by (1) for some given  $\mathbf{x}_0$ .

**A condition for a system to be conservative.** Consider the system (1). How do we check if the system is conservative, i.e., if there exists a scalar field  $E(\mathbf{x})$  that is preserved along trajectories? By definition  $E(\mathbf{x})$  is preserved along trajectories if and only if

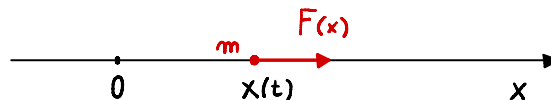
$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \nabla E(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt} = 0 \quad \Rightarrow \quad \nabla E(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = 0. \quad (3)$$

Hence, a system is conservative if and only if we can find a function  $E(\mathbf{x})$  the gradient of which is orthogonal to  $\mathbf{f}(\mathbf{x})$  at each point  $\mathbf{x}$ . In other words, the system (1) is conservative if there exists a solution to the following partial differential equation (PDE)

$$\nabla E(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = 0, \quad (4)$$

where  $E(\mathbf{x})$  is unknown and  $\mathbf{f}(\mathbf{x})$  is the vector field at the right hand side of (1).

**Point mass moving on a line.** Consider a point particle moving on a straight line under the action of an external force  $F(x)$  depending only on the position  $x$  of the particle.



The equation of motion is the Newton's second law

$$m \frac{d^2x}{dt^2} = F(x), \quad (5)$$

where  $m$  is the mass of the particle. We have seen in the course note 1 that  $F(x)$  can be always written in terms of a *potential energy function*  $V(x)$  as

$$F(x) = -\frac{dV(x)}{dx}. \quad (6)$$

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<sup>1</sup>The constant depend on the particular trajectory, i.e., on the initial condition  $\mathbf{x}_0$  (see Eq. (2)).

Multiplying (5) by the velocity  $dx(t)/dt$  of the particle and collecting terms yields

$$\frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + V(x) \right] = 0. \quad (7)$$

Hence, the quantity

$$\underbrace{E \left( x, \frac{dx}{dt} \right)}_{\text{total energy}} = \underbrace{\frac{1}{2} m \left( \frac{dx}{dt} \right)^2}_{\text{kinetic energy}} + \underbrace{V(x)}_{\text{potential energy}}, \quad (8)$$

is conserved along trajectories of (5). Upon definition of  $x_1(x) = x(t)$  and  $x_2(t) = dx(t)/dt$ , we can re-write (5) as a two-dimensional system of first-order ODEs

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{1}{m} \frac{dV(x_1)}{dx_1} \end{cases} \quad (9)$$

This allows us to write the energy function (8) as

$$E(x_1, x_2) = \frac{1}{2} m x_2^2 + V(x_1). \quad (10)$$

Note that the gradient of energy function  $E(x_1, x_2)$  is orthogonal to the vector field at the right hand side of (9). In fact,

$$\nabla E(x_1, x_2) \cdot \mathbf{f}(\mathbf{x}) = \left( \frac{dV}{dx_1}, m x_2 \right) \cdot \left( x_2, -\frac{1}{m} \frac{dV}{dx_1} \right) = \frac{dV}{dx_1} x_2 - \frac{dV}{dx_1} x_2 = 0. \quad (11)$$

Therefore condition (4) is satisfied.

**Duffing oscillator.** The Duffing equation is a non-linear second-order differential equation that models certain oscillators. In its simplest form the equation can be written as <sup>2</sup>

$$\frac{d^2 x}{dt^2} = x - x^3, \quad (13)$$

The “potential energy” corresponding to  $x - x^3$  is

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \quad (14)$$

We rewrite the system (13) as

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_1 - x_1^3 \end{cases} \quad (15)$$

<sup>2</sup>The full form of the Duffing equation has a friction term proportional to  $\dot{x}$  and a sinusoidal driving force

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t) \quad (12)$$

This system can exhibit very complex dynamics, including chaotic phase similarities and recurrences (see, e.g., C. Bonatto, J. A. C. Gallas, and Y. Ueda, “Chaotic phase similarities and recurrences in a damped-driven Duffing oscillator” *Phys. Rev. E*, **77**, 026217 (2008)).

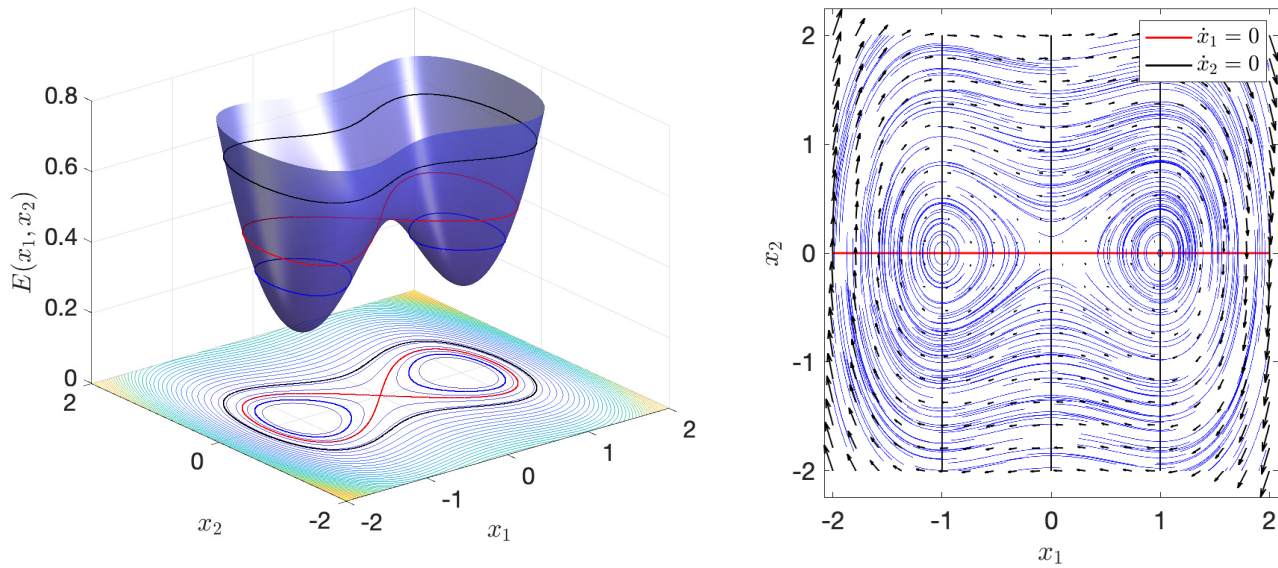


Figure 1: The Duffing oscillator (13) is a conservative system. Shown are the energy function (17) and some of its level sets. Such level sets are trajectories of the system (15). This means that we can construct the phase portrait of the system by taking sections of the energy function and then projecting such level sets onto the  $(x_1, x_2)$  plane. The Duffing oscillator has three fixed points: one saddle node at  $(x_1, x_2) = (0, 0)$  and two centers at  $(x_1, x_2) = (\pm 1, 0)$ .

Multiplying the second equation by  $x_2$  yields

$$x_2 \frac{dx_2}{dt} = (x_1 - x_1^3)x_2 = -\frac{dV(x_1)}{dt} \quad \Rightarrow \quad \frac{d}{dt} \left[ \frac{x_2^2}{2} + V(x_1) \right] = 0 \quad (16)$$

Hence, the following quantity is conserved along the trajectories

$$E(x_1, x_2) = \frac{1}{2} (x_2^2 - x_1^2) + \frac{x_1^4}{4}, \quad (17)$$

i.e., trajectories are level sets of  $E(x_1, x_2)$ . As before the gradient of  $E(x_1, x_2)$  satisfies the orthogonality condition (4). In Figure 1 we demonstrate the correspondence between the levels sets of the energy function (17) and trajectories of the system (15).

**Nonlinear pendulum.** Consider the pendulum sketched in Figure 2. We know from classical mechanics that the equation of motion is (see the course note 3 for a derivation)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin(\theta). \quad (18)$$

As before, set  $x_1(t) = \theta(t)$  and  $x_2(t) = d\theta(t)/dt$ . This allows us to write the second-order nonlinear ODE (18) as a two-dimensional dynamical system

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{g}{L} \sin(x_1) \end{cases} \quad (19)$$

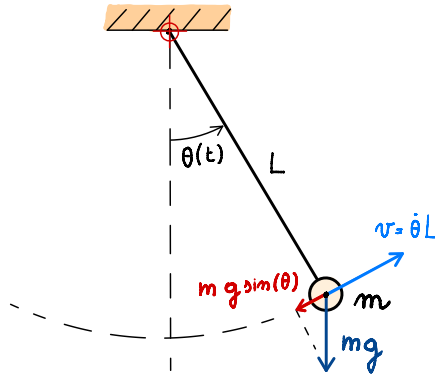


Figure 2: Sketch of a pendulum. The pendulum is assumed to have no friction, i.e., the only external force acting on the point mass  $m$  is gravity.

The “potential energy” corresponding to the force  $F(x) = -\frac{g}{L} \sin(x_1)$  is (modulus an arbitrary constant)

$$V(x_1) = -\frac{g}{L} \cos(x_1). \quad (20)$$

Multiplying the second equation in (19) by  $x_2$  we obtain

$$x_2 \frac{dx_2}{dt} = \frac{g}{L} \frac{d \cos(x_1)}{dt} \quad \Rightarrow \quad \frac{d}{dt} \left[ \frac{x_2^2}{2} - \frac{g}{L} \cos(x_1) \right] = 0 \quad (21)$$

Therefore the following quantity (total energy)

$$E(x_1, x_2) = \frac{1}{2} x_2^2 - \frac{g}{L} \cos(x_1) \quad (22)$$

is preserved along any trajectory of the system (19). This implies that the trajectories of the pendulum in the phase space  $(x_1, x_2)$  (cylindrical phase space) are level sets of (22) (see Figure 3). As before the gradient of  $E(x_1, x_2)$  satisfies the orthogonality condition (4).

**Dynamics of a point mass in three-dimensional space.** Consider Newton’s second law

$$m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}) \quad (23)$$

describing the dynamics of a point mass  $m$  subject to the external force  $\mathbf{F}(\mathbf{x})$ . Such force depends only on the position of the particle, and it could be due to gravity, electromagnetic fields, etc. If the force  $\mathbf{F}(\mathbf{x})$  is conservative (see Appendix A for a review of conservative vector fields), then there exists a potential energy  $V(\mathbf{x})$  such that

$$\mathbf{F}(\mathbf{x}) = -\nabla V(\mathbf{x}). \quad (24)$$

This allows us to write equation (23) as

$$m \frac{d^2 \mathbf{x}}{dt^2} = -\nabla V(\mathbf{x}). \quad (25)$$

By taking the scalar (dot) product of the vector equation (23) by  $d\mathbf{x}/dt$  we obtain

$$m \frac{d\mathbf{x}}{dt} \cdot \frac{d^2 \mathbf{x}}{dt^2} = m \left( \frac{dx_1}{dt} \frac{d^2 x_1}{dt^2} + \frac{dx_2}{dt} \frac{d^2 x_2}{dt^2} + \frac{dx_3}{dt} \frac{d^2 x_3}{dt^2} \right) = -\frac{dx_1}{dt} \frac{\partial V}{\partial x_1} - \frac{dx_2}{dt} \frac{\partial V}{\partial x_2} - \frac{dx_3}{dt} \frac{\partial V}{\partial x_3}, \quad (26)$$

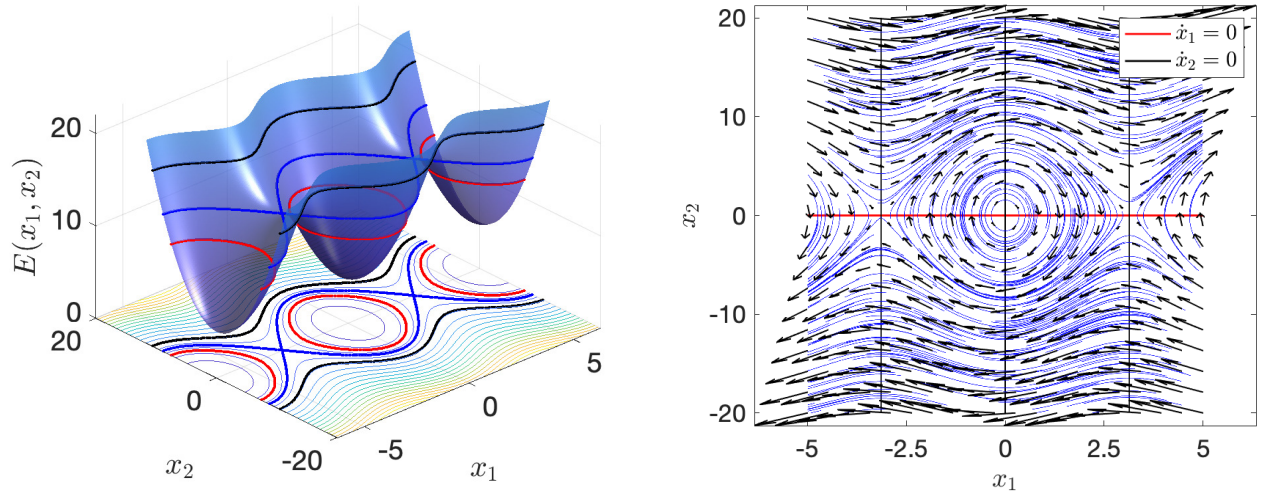


Figure 3: Nonlinear pendulum (18) for  $g = 9.8 \text{ m/s}^2$ , and  $L = 0.5 \text{ m}$ . Shown are the energy function (22) and some of its level sets. Such level sets are trajectories of the system (18). This means that we can construct the phase portrait of the system by taking sections of the energy function and projecting the corresponding level sets onto the  $(x_1, x_2)$  plane. The pendulum has two fixed points repeating periodically (with period  $2\pi$ ): a saddle node at  $(x_1, x_2) = (\pi, 0)$  and a center at  $(x_1, x_2) = (0, 0)$ .

i.e.,

$$\frac{d}{dt} \left( \frac{1}{2} m \|\dot{\mathbf{x}}\|^2 + V(\mathbf{x}) \right) = 0. \quad (27)$$

Once again, the total energy (sum of kinetic and potential energy)

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} m \|\dot{\mathbf{x}}\|_2^2 + V(\mathbf{x}) \quad (28)$$

is conserved along trajectories if  $\mathbf{F}(\mathbf{x})$  in (23) is a potential vector field.

**Dynamics of a point mass on a surface.** Consider a point mass moving without friction on a surface  $S(x_1, x_2)$  under the effect of gravity (Figure 4). The coordinates of the point mass in the three dimensional space are

$$(x_1(t), x_2(t), S(x_1(t), x_2(t))). \quad (29)$$

This implies that the velocity  $\mathbf{v}(t)$  of the particle on the surface can be expressed as

$$\mathbf{v}(t) = \left( \dot{x}_1(t), \dot{x}_2(t), \frac{\partial S(x_1(t), x_2(t))}{\partial x_1} \dot{x}_1(t) + \frac{\partial S(x_1(t), x_2(t))}{\partial x_2} \dot{x}_2(t) \right) \quad (30)$$

Hence, the kinetic energy of the particle is

$$\begin{aligned} T(\dot{x}_1, \dot{x}_2, x_1, x_2) = & \frac{1}{2} m \left[ \dot{x}_1^2 + \dot{x}_2^2 + \left( \frac{\partial S(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial S(x_1, x_2)}{\partial x_2} \dot{x}_2 \right)^2 \right] \\ & \frac{1}{2} m \left[ \dot{x}_1^2 \left( 1 + \frac{\partial S(x_1, x_2)}{\partial x_1} \right) + \dot{x}_2^2 \left( 1 + \frac{\partial S(x_1, x_2)}{\partial x_2} \right) + 2\dot{x}_1\dot{x}_2 \frac{\partial S(x_1, x_2)}{\partial x_1} \frac{\partial S(x_1, x_2)}{\partial x_2} \right]. \end{aligned} \quad (31)$$

On the other hand, the potential energy due to gravity is

$$V(x_1, x_2) = mgS(x_1, x_2). \quad (32)$$

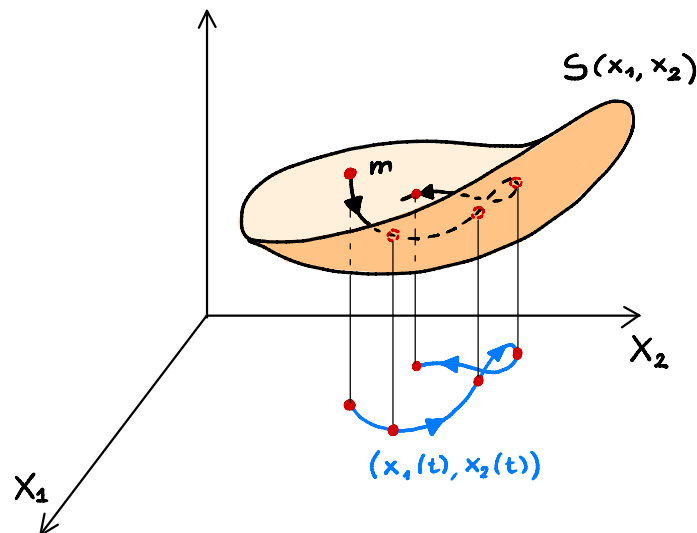


Figure 4: Dynamics of a point mass on a surface  $S(x_1, x_2)$  under the effect of the gravity.

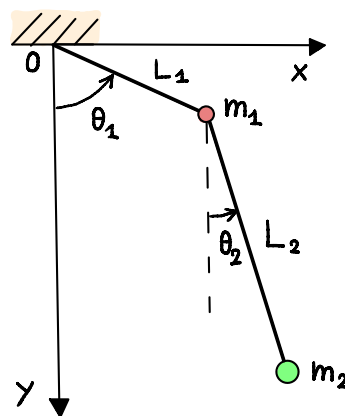


Figure 5: Sketch of a double pendulum. Gravity acts on both masses. There is also an interaction between  $m_1$  and  $m_2$  through the rod of length  $L_2$ .

The total energy of the system is

$$E(\dot{x}_1, \dot{x}_2, x_1, x_2) = T(\dot{x}_1, \dot{x}_2, x_1, x_2) + V(x_1, x_2), \quad (33)$$

where  $T$  and  $V$  are given in (31) and (32). Since there is no friction  $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$  is conserved along trajectories, i.e., the system is conservative. Hence, the trajectories are level sets of  $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$ . In other words, the relation between the coordinates  $(x_1, x_2)$  and velocity  $(\dot{x}_1, \dot{x}_2)$  of the point mass  $m$  is fully determined by the level sets of  $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$ . The value of the level set depends on the initial condition of the particle (position and velocity). In the case where the surface  $S$  is a sphere, this system is called *spherical pendulum*. In the next course note on Lagrangian and Hamiltonian dynamics we will see how to derive the equations of motion for this system.

**Double pendulum.** Consider the double pendulum sketched in Figure 5 We can express the position

of the point masses  $m_1$  and  $m_2$  relative to the coordinate system  $(x, y)$  as

$$\begin{cases} x_1 = L_1 \sin(\theta_1) \\ y_1 = L_1 \cos(\theta_1) \end{cases} \quad \begin{cases} x_2 = x_1 + L_2 \sin(\theta_2) \\ y_2 = y_1 + L_2 \cos(\theta_2) \end{cases} \quad (34)$$

The total potential energy is the sum of the potential gravitational energies of both masses

$$V(\theta_1, \theta_2) = (m_1 + m_2)gL_1(1 - \cos(\theta_1)) + m_2gL_2(1 - \cos(\theta_2)). \quad (35)$$

On the other hand, the kinetic energy is

$$\begin{aligned} T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right). \end{aligned} \quad (36)$$

Note that the kinetic energy is a quadratic form in  $\dot{\theta}_1$  and  $\dot{\theta}_2$ . In this case, the total energy of the system can be written as

$$\begin{aligned} E(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) + V(\theta_1, \theta_2) \\ &= \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) + \\ &\quad (m_1 + m_2)gL_1(1 - \cos(\theta_1)) + m_2gL_2(1 - \cos(\theta_2)). \end{aligned} \quad (37)$$

Remarkably, the trajectories of the double pendulum can be chaotic. Yet, they are level sets of the four-dimensional energy function (37). In the next course note on Lagrangian and Hamiltonian dynamics we will see how to derive the equations of motion for this system.

**Properties of conservative systems.** Trajectories of conservative systems are level sets of some energy function  $E(\mathbf{x})$  (see Equation (2)). Based on this observation it is straightforward to prove the following properties

- Relative maxima and relative minima of the energy function  $E(\mathbf{x})$  are fixed points of the system. Moreover, trajectories nearby such fixed points are necessarily closed. In two dimensions maxima and minima of the energy function  $E(\mathbf{x})$  are *centers* (see Figure 1 and Figure 3). In higher dimensions such local maxima and minima define a closed trajectory on a high-dimensional torus.
- Conservative systems cannot have attractors or repellers. This includes stable/unstable nodes, spirals, limit cycles, and any other attracting or repelling set (e.g., fractal attractors).
- Saddle nodes in 2D conservative system are saddle points of the energy function. This is clearly demonstrated in Figure 1 and Figure 3.

**Gradient systems.** We say that the  $n$ -dimensional dynamical system (1) is a gradient system if it can be written as

$$\frac{d\mathbf{x}}{dt} = -\nabla V(\mathbf{x}). \quad (38)$$

In other words, a gradient system is a first-order dynamical system in which the vector field  $\mathbf{f}(\mathbf{x})$  is conservative<sup>3</sup> (see Appendix A for a review of conservative vector fields). The function  $V(\mathbf{x})$  in (38) is called potential energy of  $\mathbf{f}(\mathbf{x})$ . Note that  $V(\mathbf{x})$  is the opposite of the potential function  $\varphi(\mathbf{x})$  discussed

<sup>3</sup>It is important to distinguish second-order dynamical systems with potential forces (e.g., (25)) from first-order gradient system of the form (38). Gradient systems are, in general, *not* conservative.

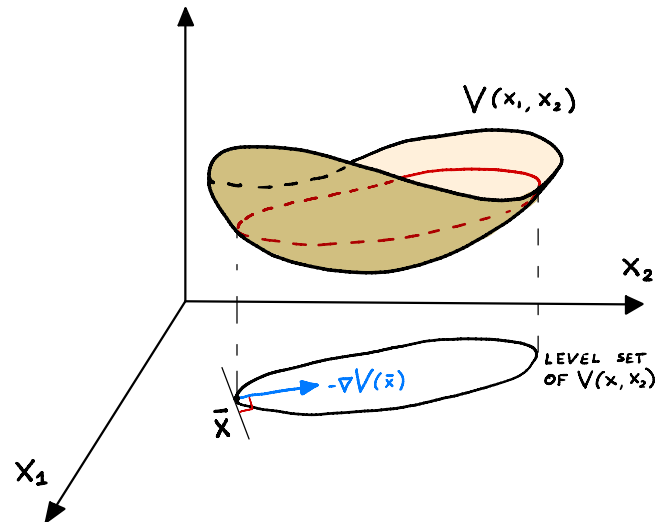


Figure 6: Geometric meaning of the gradient  $-\nabla V(x_1, x_2)$ .

in Appendix A. To check if a dynamical system is a gradient system, we simply need to check if  $\mathbf{f}(\mathbf{x})$  is conservative (conditions given in Appendix A).

As shown in Figure 6, the vector  $-\nabla V(\mathbf{x})$  is orthogonal to the level set of  $V(\mathbf{x})$  passing through  $\mathbf{x}$ , and it points in the direction where  $V(\mathbf{x})$  decreases the most. To prove orthogonality, consider a point  $\mathbf{x}$  and parameterize the level set  $V(\mathbf{x})$  via a curve  $\mathbf{s}(\lambda)$  such that  $\mathbf{s}(0) = \mathbf{x}$  ( $\lambda$  is the parameter of the curve). By expanding  $\mathbf{s}(\lambda)$  in a Taylor series at  $\lambda = 0$  yields

$$\mathbf{s}(\lambda) = \mathbf{s}(0) + \frac{d\mathbf{s}(0)}{d\lambda}\lambda + \dots \quad (39)$$

Evaluate  $V(\mathbf{x})$  along the curve (39) to obtain

$$V(\mathbf{s}(\lambda)) = V(\mathbf{s}(0)) + \nabla V(\mathbf{s}(0)) \cdot \frac{d\mathbf{s}(0)}{d\lambda}\lambda + \dots \quad \Rightarrow \quad \nabla V(\mathbf{x}) \cdot \frac{d\mathbf{s}(0)}{d\lambda} = 0. \quad (40)$$

This implies that  $\nabla V(\mathbf{x}) \perp d\mathbf{s}(0)/d\lambda$ , i.e., that  $\nabla V(\mathbf{x})$  is orthogonal to the level set of  $V(\mathbf{x})$  passing through  $\mathbf{x}$ . To see why, simply recall that the vector  $d\mathbf{s}(0)/d\lambda$  is tangent to the curve  $\mathbf{s}(\lambda)$  at  $\mathbf{s}(0) = \mathbf{x}$ . On the other hand, trajectories of conservative systems are level sets of some energy function.

Gradient systems arise very often in optimization theory, e.g., when computing the (relative) minimum of a function  $V(\mathbf{x})$ . In this context, (38) represents a continuous version of the so-called *gradient descent* scheme, in which the relative minimum of a cost function  $V(\mathbf{x})$  is computed iteratively via

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \gamma \nabla V(\mathbf{x}_k), \quad (41)$$

where  $\gamma > 0$  is a real number called “learning rate”.

Clearly, the critical points of  $V(\mathbf{x})$ , i.e., maxima, minima, saddles, and any other point  $\mathbf{x}^*$  satisfying

$$\nabla V(\mathbf{x}^*) = \mathbf{0}$$

are fixed points of the system (38). Moreover,  $V(\mathbf{x})$  decreases monotonically along trajectories of the system (38) (recall that  $\nabla V$  is orthogonal to the level sets of  $V$ ). This can be also shown by a direct



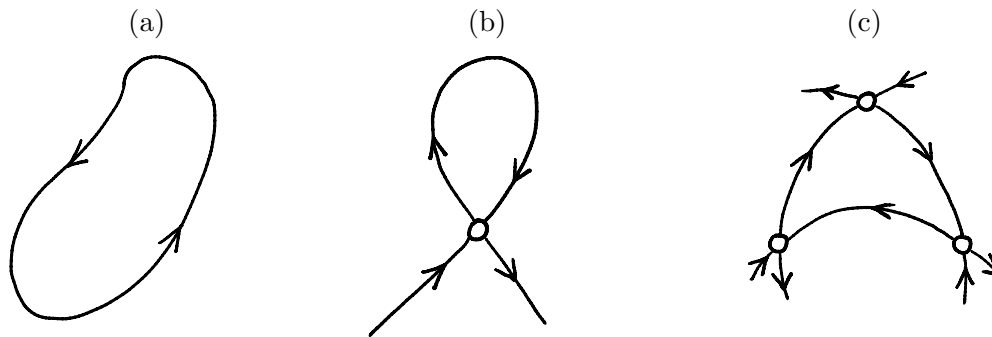


Figure 7: Gradient systems cannot have limit cycles (a), homoclinic orbits (b), or heteroclinic cycles (c).

calculation,

$$\begin{aligned}
 V(\mathbf{x}(t_2)) &= V(\mathbf{x}(t_1)) + \int_{t_1}^{t_2} \frac{dV(\mathbf{x}(t))}{dt} dt \quad t_2 \geq t_1 \\
 &= V(\mathbf{x}(t_1)) + \int_{t_1}^{t_2} \nabla V(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}(s)}{dt} dt \\
 &= V(\mathbf{x}(t_1)) - \int_{t_1}^{t_2} \nabla V(\mathbf{x}(t)) \cdot \nabla V(\mathbf{x}(t)) dt \\
 &= V(\mathbf{x}(t_1)) - \int_{t_1}^{t_2} \|\nabla V(\mathbf{x}(t))\|_2^2 dt \\
 &\leq V(\mathbf{x}(t_1)),
 \end{aligned} \tag{42}$$

where the equality sign holds if and only if  $\|\nabla V(\mathbf{x}(t))\|_2 = 0$ , i.e., only at fixed points of (38). Equation (42) rules out the possibility of any closed orbit in gradient systems, including limit cycles, homoclinic orbits, and heteroclinic cycles (see Figure 7).

The Jacobian matrix of a gradient system is necessarily *symmetric* (see Eq. (44)). Therefore it can only have *real eigenvalues*. This implies that the flow of a gradient system nearby a hyperbolic fixed points cannot have any rotating component due to complex conjugate eigenvalues, i.e., no spirals. In other words, hyperbolic fixed points of gradient systems can only be sources, sinks, or saddle nodes!

## Appendix A: Conservative vector fields

Let  $\mathbf{F}(\mathbf{x})$  be a continuously differentiable vector field defined in a *simply connected* domain<sup>4</sup>  $D \subseteq \mathbb{R}^n$ . We say that  $\mathbf{F}(\mathbf{x})$  is conservative if it can be written as a gradient of some function  $\varphi : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , i.e.,

$$\mathbf{F}(\mathbf{x}) = \nabla \varphi(\mathbf{x}). \tag{43}$$

There are several equivalent conditions that characterize conservative vector fields in a simply connected domain:

a) The Jacobian of  $\mathbf{F}(\mathbf{x})$  is symmetric, i.e.,

$$\frac{\partial F_i(\mathbf{x})}{\partial x_j} = \frac{\partial F_j(\mathbf{x})}{\partial x_i} \quad \text{for all } \mathbf{x} \in D \text{ and for all } i, j = 1, \dots, n. \tag{44}$$

In two and three dimensions this condition is equivalent to the well known condition expressing the fact that  $\mathbf{F}(\mathbf{x})$  is “irrotational” or curl-free.

<sup>4</sup>A domain  $D \subseteq \mathbb{R}^n$  is called simply connected if any closed curve in  $D$  can be shrunk to a point without getting out of  $D$ .

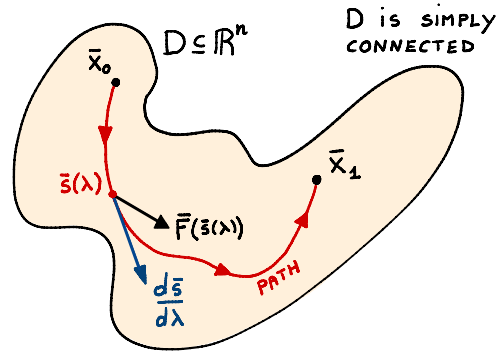


Figure 8: Sketch of a simply connected domain  $D \subseteq \mathbb{R}^n$ . Shown are two points  $\mathbf{x}_0, \mathbf{x}_1 \in D$ , a path  $\mathbf{s}(\lambda)$  connecting  $\mathbf{x}_0 = \mathbf{s}(0)$  to  $\mathbf{x}_1 = \mathbf{s}(1)$  the vector field  $\mathbf{F}(\mathbf{x})$  along the path, and the velocity vector  $d\mathbf{s}(\lambda)/d\lambda$ . If  $\mathbf{F}(\mathbf{x})$  is conservative, then the line integral (45) does not depend on the path connecting  $\mathbf{x}_0$  to  $\mathbf{x}_1$ .

b) The line integral of  $\mathbf{F}(\mathbf{x})$  along a smooth curve  $\mathbf{s}(\lambda)$  in  $D$  connecting two arbitrary points  $\mathbf{x}_0, \mathbf{x}_1 \in D$

$$\varphi(\mathbf{x}_1) = \varphi(\mathbf{x}_0) + \int_0^1 \mathbf{F}(\mathbf{s}(\lambda)) \cdot \frac{d\mathbf{s}(\lambda)}{d\lambda} d\lambda \tag{45}$$

does not depend on the path connecting  $\mathbf{x}_0$  to  $\mathbf{x}_1$  (see Figure 8).

c) The line integral of  $\mathbf{F}(\mathbf{x})$  along any smooth closed curve<sup>5</sup>  $\mathbf{s}(\lambda)$  is zero, i.e.,

$$\int_0^1 \mathbf{F}(\mathbf{s}(\lambda)) \cdot \frac{d\mathbf{s}(\lambda)}{d\lambda} d\lambda = 0 \tag{46}$$

This follows immediately from (46) if we set  $\mathbf{x}_1 = \mathbf{x}_0$ .

It can be shown that (43), (44), and path independence of (45) all imply each other. For example, if the symmetry condition (44) is satisfied for all  $\mathbf{x} \in D$  ( $D$  simply connected) then there exists a potential  $\varphi(\mathbf{x})$  satisfying (43).

The potential  $\varphi(\mathbf{x})$  can be determined by computing the line integral (45) along *any* path connecting two points in  $D$ . In particular, if  $D$  is convex we can consider a simple line as a path

$$\mathbf{s}(\lambda) = (1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}_1 \quad \lambda \in [0, 1]. \tag{47}$$

*Example:* Consider the two dimensional vector field  $\mathbf{F}(\mathbf{x}) = (F_1(x_1, x_2), F_2(x_1, x_2))$  where

$$F_1(x_1, x_2) = 2x_1x_2, \quad F_2(x_1, x_2) = x_1^2. \tag{48}$$

It is straightforward to show that  $\mathbf{F}(\mathbf{x})$  conservative. In fact, the Jacobain of  $\mathbf{F}$  is symmetric for all  $\mathbf{x} \in \mathbb{R}^2$

$$\frac{\partial F_1(x_1, x_2)}{\partial x_2} = 2x_1 = \frac{\partial F_2(x_1, x_2)}{\partial x_1}. \tag{49}$$

Therefore the symmetry condition (44) is satisfied. To compute a potential  $\varphi(\mathbf{x})$  consider a line (47) connecting an arbitrary point  $\mathbf{x}$  to the origin, i.e.,

$$\mathbf{s}(\lambda) = \lambda\mathbf{x} \quad \frac{d\mathbf{s}(\lambda)}{d\lambda} = \mathbf{x}. \tag{50}$$

<sup>5</sup>A closed curve satisfies  $\mathbf{x}_0 = \mathbf{s}(0) = \mathbf{s}(1) = \mathbf{x}_1$ .

We are free to choose any curve we like to evaluate the integral (45) (we already know that the vector field is conservative). We have,

$$\begin{aligned}
 \varphi(\mathbf{x}) &= \varphi(\mathbf{0}) + \int_0^1 \mathbf{F}(\mathbf{s}(\lambda)) \cdot \frac{d\mathbf{s}(\lambda)}{d\lambda} d\lambda \\
 &= \varphi(\mathbf{0}) + \int_0^1 \mathbf{F}(\lambda\mathbf{x}) \cdot \mathbf{x} d\lambda \\
 &= \varphi(\mathbf{0}) + \int_0^1 (2\lambda^2 x_1^2 x_2 + \lambda^2 x_1^2 x_2) d\lambda \\
 &= \varphi(\mathbf{0}) + x_1^2 x_2.
 \end{aligned} \tag{51}$$

The additive constant  $\varphi(\mathbf{0})$  does not change the gradient representing  $\mathbf{F}(\mathbf{x})$  in (43). Therefore we can set it to zero. This yields the potential

$$\varphi(\mathbf{x}) = x_1^2 x_2. \tag{52}$$

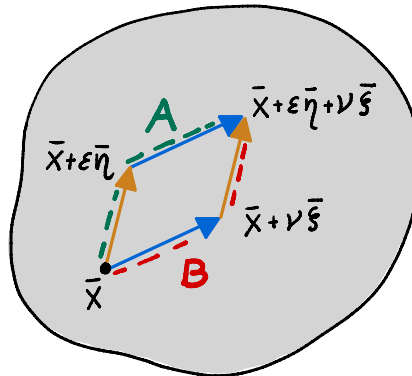
Let us verify that the gradient of (52) coincides with  $\mathbf{F}(\mathbf{x})$ . We have

$$\nabla\varphi(\mathbf{x}) = \left( \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2} \right). \tag{53}$$

yields the vector field (48). We have,

$$\begin{aligned}
 \frac{\partial\varphi}{\partial x_1} &= 2x_1 x_2 && \text{(same as } F_1(x_1, x_2)), \\
 \frac{\partial\varphi}{\partial x_2} &= x_1^2 && \text{(same as } F_2(x_1, x_2)).
 \end{aligned}$$

**Path invariance implies symmetry of the Jacobian.** It is instructive to show how (44) follows directly from (45). To this end, consider a point  $\mathbf{x}$  and its infinitesimal neighborhood. All points in such neighborhood can be connected via two straight lines as follows



By imposing that the line integral of  $\mathbf{F}(\mathbf{x})$  along the paths A and B above are identical we obtain<sup>6</sup>

$$\epsilon \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\eta} + \nu \mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta}) \cdot \boldsymbol{\xi} = \nu \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\xi} + \epsilon \mathbf{F}(\mathbf{x} + \nu\boldsymbol{\xi}) \cdot \boldsymbol{\xi}. \tag{56}$$

<sup>6</sup>Note that along the infinitesimal path (line)  $\mathbf{s}(\epsilon) = \mathbf{x} + \epsilon\boldsymbol{\eta}$

$$\int_0^\epsilon \mathbf{F}(\mathbf{s}(\epsilon)) \cdot \frac{d\mathbf{s}(\epsilon)}{d\epsilon} d\epsilon = \epsilon \mathbf{F}(\mathbf{x}) \cdot \boldsymbol{\eta}. \tag{54}$$

Similarly, along the path  $\mathbf{s}(\nu) = \mathbf{x} + \epsilon\boldsymbol{\eta} + \nu\boldsymbol{\xi}$

$$\int_0^\nu \mathbf{F}(\mathbf{s}(\nu)) \cdot \frac{d\mathbf{s}(\nu)}{d\nu} d\nu = \nu \mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta}) \cdot \boldsymbol{\xi}. \tag{55}$$

This equation can be simplified further by expanding  $\mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta})$  and  $\mathbf{F}(\mathbf{x} + \nu\boldsymbol{\xi})$  in a first-order Taylor series, i.e.,

$$\begin{aligned}\mathbf{F}(\mathbf{x} + \epsilon\boldsymbol{\eta}) &\simeq \mathbf{F}(\mathbf{x}) + \epsilon\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\eta}, \\ \mathbf{F}(\mathbf{x} + \nu\boldsymbol{\xi}) &\simeq \mathbf{F}(\mathbf{x}) + \nu\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\xi}.\end{aligned}\tag{57}$$

A substitution of these expansions into (56) yields

$$[\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\eta}] \cdot \boldsymbol{\xi} = [\mathbf{J}_{\mathbf{F}}(\mathbf{x})\boldsymbol{\xi}] \cdot \boldsymbol{\eta}.\tag{58}$$

This condition implies (for  $\boldsymbol{\xi}$  and  $\boldsymbol{\eta}$  arbitrary) that the Jacobian of  $\mathbf{F}(\mathbf{x})$  must be a symmetric matrix at  $\mathbf{x}$ .