Conservative and gradient systems

Conservative systems. consider the n-dimensional nonlinear dynamical system

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{1}$$

We say that the system is *conservative* if there exists a scalar field $E(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$ that remains constant[1] along each trajectory of (1), and is not constant on any open subset $D \subseteq \mathbb{R}^n$. This definition implies that the trajectories of a conservative system are level sets of the scalar field $E(\boldsymbol{x})$, i.e.,

$$E\big(\boldsymbol{X}(t, \boldsymbol{x}_0)\big) = E(\boldsymbol{x}_0), \tag{2}$$

where $\boldsymbol{X}(t, \boldsymbol{x}_0)$ denotes a trajectory of (1) corresponding to the initial condition $\boldsymbol{x}_0$.

A condition for a system to be conservative. How can we determine whether the system(1) is conservative, that is, whether there exists a scalar field $E(\boldsymbol{x})$ that is preserved along trajectories? By definition, the quantity $E(\boldsymbol{x})$ is conserved along solutions if and only if

$$\frac{dE}{dt} = 0 \quad \Longrightarrow \quad \nabla E(\boldsymbol{x}) \cdot \frac{d\boldsymbol{x}}{dt} = 0 \quad \Longrightarrow \quad \nabla E(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) = 0. \tag{3}$$

Hence, the system is conservative if and only if there exists a scalar function $E(\boldsymbol{x})$ whose gradient is orthogonal to the vector field $\boldsymbol{f}(\boldsymbol{x})$ at every point $\boldsymbol{x}$. In other words, the system (1) is conservative if there exists a "global" solution to the following first-order partial differential equation (PDE)

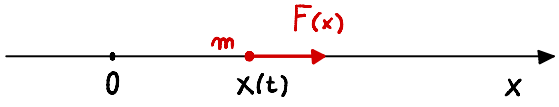
$$\nabla E(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) = 0, \tag{4}$$

where $E(\boldsymbol{x})$ is the unknown scalar function, and $\boldsymbol{f}(\boldsymbol{x})$ is the vector field appearing on the right-hand side of (1).

Remark: We can always find a local function of (4) i.e., a function $E(\boldsymbol{x})$ that solves (4) in a neighborhood of some point $\boldsymbol{x}_0$. To this end, it is sufficient to consider any $(n-1)$-dimensional Poincaré section (a surface transverse to the flow) in a neighborhood of a point $\boldsymbol{x}_0$, and define $E(\boldsymbol{x})$ for all $\boldsymbol{x}$ on that section and sufficiently close to $\boldsymbol{x}_0$. The values of E can then be extended to a neighborhood of $\boldsymbol{x}_0$ by advecting the surface along the flow of the system for some time. Since E remains constant along trajectories by construction, this procedure defines a local solution to the PDE (4). In general,

A few examples of conservative systems

Point mass moving on a line. Consider a point particle moving on a straight line under the action of an external force $F(x)$ depending only on the position x of the particle.



[1] The value of E depends on the specific trajectory, i.e., on the initial condition $\boldsymbol{x}_0$(see Eq. (2)).

The equation of motion is the Newton's second law

$$m\frac{d^2x}{dt^2} = F(x), \tag{5}$$

where m is the mass of the particle. Clearly, $F(x)$ can be always written in terms of a *potential energy function* $V(x)$ as

$$F(x) = -\frac{dV(x)}{dx}. \tag{6}$$

Multiplying (5) by the velocity $dx(t)/dt$ of the particle and collecting terms yields

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x)\right] = 0. \tag{7}$$

Hence, the quantity

$$\underbrace{E\left(x, \frac{dx}{dt}\right)}_{\text{total energy}} = \underbrace{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2}_{\text{kinetic energy}} + \underbrace{V(x)}_{\text{potential energy}}, \tag{8}$$

is conserved along trajectories of (5). Upon definition of $x_1(x) = x(t)$ and $x(t) = dx(t)/dt$, we can re-write (5) as a two-dimensional system of first-order ODEs

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{1}{m}\dfrac{dV(x_1)}{dx_1} \end{cases} \tag{9}$$

This allows us to write the energy function (8) as

$$E(x_1, x_2) = \frac{1}{2}mx_2^2 + V(x_1). \tag{10}$$

Note that the gradient of energy function $E(x_1, x_2)$ is orthogonal to the vector field at the right hand side of (9). In fact,

$$\nabla E(x_1, x_2) \cdot \boldsymbol{f}(\boldsymbol{x}) = \left(\frac{dV}{dx_1}, mx_2\right) \cdot \left(x_2, -\frac{1}{m}\frac{dV}{dx_1}\right) = \frac{dV}{dx_1}x_2 - \frac{dV}{dx_1}x_2 = 0. \tag{11}$$

Therefore condition (4) is satisfied.

Duffing oscillator. The Duffing equation is a non-linear second-order differential equation that models certain oscillators. In its simplest form the equation can be written as [2]

$$\frac{d^2x}{dt^2} = x - x^3, \tag{13}$$

The "potential energy" corresponding to $x - x^3$ is

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \tag{14}$$

[2] The full form of the Duffing equation has a friction term proportional to $\dot{x}$ and a sinusoidal driving force

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t) \tag{12}$$

This system can exhibit very complex dynamics, including chaotic phase similarities and recurrences (see, e.g., C. Bonatto, J. A. C. Gallas, and Y. Ueda, "Chaotic phase similarities and recurrences in a damped-driven Duffing oscillator" *Phys. Rev. E*, **77**, 026217 (2008).

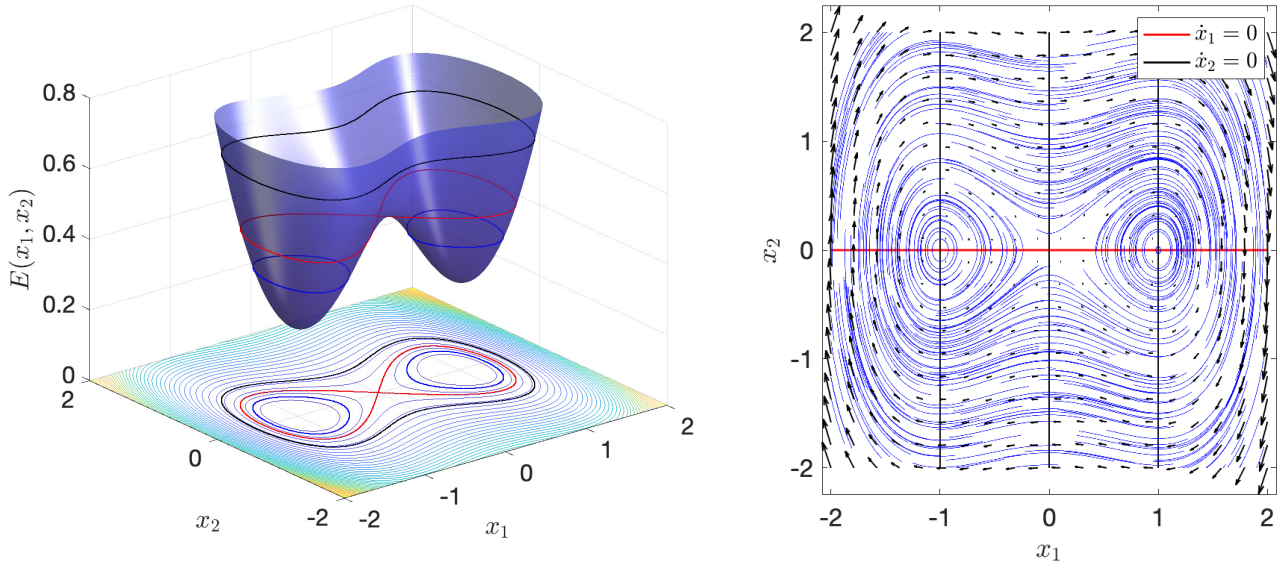


Figure 1: The Duffing oscillator (13) is a conservative system. Shown are the energy function (17) and some of its level sets. Such level sets are trajectories of the system (15). This means that we can construct the phase portrait of the system by taking sections of the energy function and then projecting such level sets onto the (x_1, x_2) plane. The Duffing oscillator has three fixed points: one saddle node at $(x_1, x_2) = (0, 0)$ and two centers at $(x_1, x_2) = (\pm 1, 0)$.

We rewrite the system (13) as

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_1 - x_1^3 \end{cases} \tag{15}$$

Multiplying the second equation by x_2 yields

$$x_2 \frac{dx_2}{dt} = (x_1 - x_1^3) x_2 = -\frac{dV(x_1)}{dt} \qquad \Rightarrow \qquad \frac{d}{dt}\left[\frac{x_2^2}{2} + V(x_1)\right] = 0 \tag{16}$$

Hence, the following quantity is conserved along the trajectories

$$E(x_1, x_2) = \frac{1}{2}\left(x_2^2 - x_1^2\right) + \frac{x_1^4}{4}, \tag{17}$$

i.e., trajectories are level sets of $E(x_1, x_2)$. As before the gradient of $E(x_1, x_2)$ satisfies the orthogonality condition (4). In Figure 1 we demonstrate the correspondence between the levels sets of the energy function (17) and trajectories of the system (15).

Nonlinear pendulum. Consider the pendulum sketched in Figure 2. We know from classical mechanics that the equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta). \tag{18}$$

As before, set $x_1(t) = \theta(t)$ and $x_2(t) = d\theta(t)/dt$. This allows us to write the second-order nonlinear ODE (18) as a two-dimensional dynamical system

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{g}{L}\sin(x_1) \end{cases} \tag{19}$$

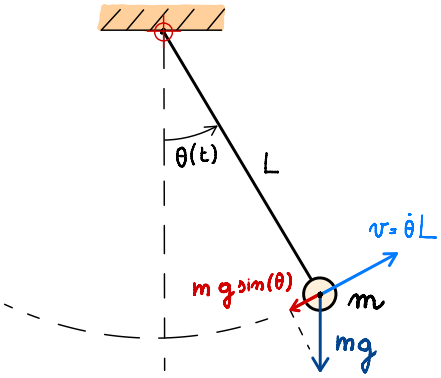


Figure 2: Sketch of a pendulum. The pendulum is assumed to have no friction, i.e., the only external force acting on the point mass m is gravity.

The "potential energy" corresponding to the force $F(x) = -\frac{g}{L}\sin(x_1)$ is (modulus an arbitrary additive constant)

$$V(x_1) = -\frac{g}{L}\cos(x_1). \tag{20}$$

Multiplying the second equation in (19) by x_2 we obtain

$$x_2\frac{dx_2}{dt} = \frac{g}{L}\frac{d\cos(x_1)}{dt} \qquad \Rightarrow \qquad \frac{d}{dt}\left[\frac{x_2^2}{2} - \frac{g}{L}\cos(x_1)\right] = 0 \tag{21}$$

Therefore the following quantity (total energy)

$$E(x_1, x_2) = \frac{1}{2}x_2^2 - \frac{g}{L}\cos(x_1) \tag{22}$$

is preserved along any trajectory of the system (19). This implies that the trajectories of the pendulum in the phase space (x_1, x_2) (cylindrical phase space) are level sets of (22) (see Figure 3). As before the gradient of $E(x_1, x_2)$ satisfies the orthogonality condition (4).

Dynamics of a point mass in three-dimensional space. Consider the dynamics of a point mass m subject to an external force field $\boldsymbol{F}(\boldsymbol{x})$. The equations of motion are obtained by applying Newton's second law:

$$m\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{F}(\boldsymbol{x}), \tag{23}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^3$ denotes the position of the particle at time t. The force $\boldsymbol{F}(\boldsymbol{x})$ is assumed to depend only on the position of the particle and may arise from, for example, gravitational or electromagnetic fields. If $\boldsymbol{F}(\boldsymbol{x})$ is a conservative vector field (see Appendix A for a review), then there exists a scalar potential energy function $V(\boldsymbol{x})$ such that

$$\boldsymbol{F}(\boldsymbol{x}) = -\nabla V(\boldsymbol{x}). \tag{24}$$

This allows us to write equation (23) as

$$m\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla V(\boldsymbol{x}). \tag{25}$$

By taking the scalar (dot) product of the vector equation (23) by $d\boldsymbol{x}/dt$ we obtain

$$m\frac{d\boldsymbol{x}}{dt}\cdot\frac{d^2\boldsymbol{x}}{dt^2} = m\left(\frac{dx_1}{dt}\frac{d^2x_1}{dt^2} + \frac{dx_2}{dt}\frac{d^2x_2}{dt^2} + \frac{dx_3}{dt}\frac{d^2x_3}{dt^2}\right) = -\frac{dx_1}{dt}\frac{\partial V}{\partial x_1} - \frac{dx_2}{dt}\frac{\partial V}{\partial x_2} - \frac{dx_3}{dt}\frac{\partial V}{\partial x_3}, \tag{26}$$

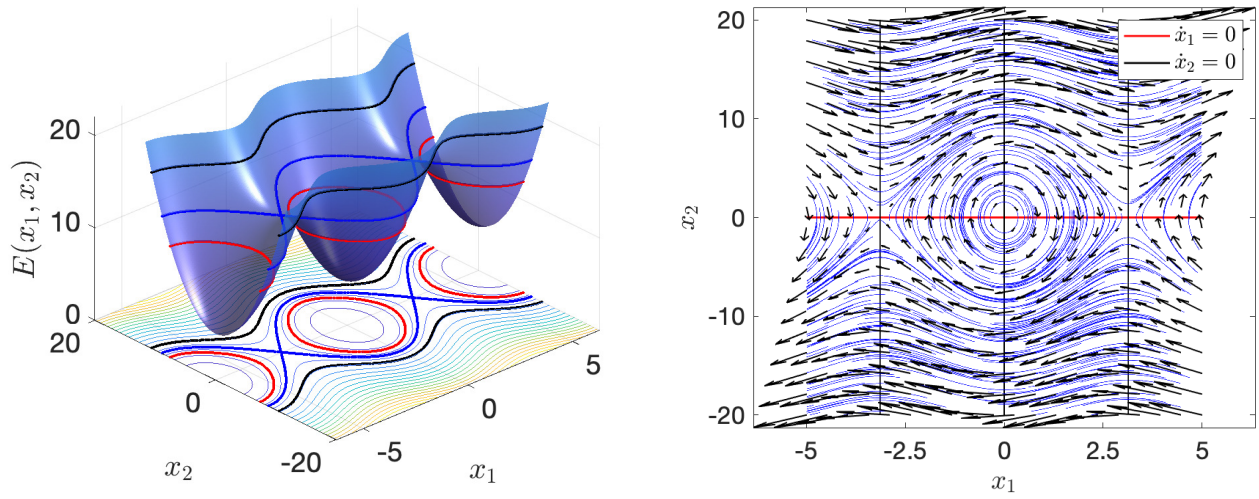


Figure 3: Nonlinear pendulum (18) for $g = 9.8$ m/s^2, and $L = 0.5$ m. Shown are the energy function (22) and some of its level sets. Such level sets are trajectories of the system (18). This means that we can construct the phase portrait of the system by taking sections of the energy function and projecting the corresponding level sets onto the (x_1, x_2) plane. The pendulum has two fixed points repeating periodically (with period 2π): a saddle node at $(x_1, x_2) = (\pi, 0)$ and a center at $(x_1, x_2) = (0, 0)$.

i.e.,

$$\frac{d}{dt}\left(\frac{1}{2}m\,\|\dot{\boldsymbol{x}}\|^2 + V(\boldsymbol{x})\right) = 0. \tag{27}$$

Once again, the total energy (sum of kinetic and potential energy)

$$E(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}m\,\|\dot{\boldsymbol{x}}\|_2^2 + V(\boldsymbol{x}) \tag{28}$$

is conserved along trajectories if $\boldsymbol{F}(\boldsymbol{x})$ in (23) is a potential vector field.

Dynamics of a point mass on a surface. Consider a point mass moving without friction on a surface $S(x_1, x_2)$ under the effect of gravity (Figure 4). The coordinates of the point mass in the three dimensional space are

$$(x_1(t), x_2(t), S(x_1(t), x_2(t)))\,. \tag{29}$$

This implies that the velocity $\boldsymbol{v}(t)$ of the particle on the surface can be expressed as

$$\boldsymbol{v}(t) = \left(\dot{x}_1(t), \dot{x}_2(t), \frac{\partial S(x_1(t), x_2(t))}{\partial x_1}\dot{x}_1(t) + \frac{\partial S(x_1(t), x_2(t))}{\partial x_2}\dot{x}_2(t)\right) \tag{30}$$

Hence, the kinetic energy of the particle is

$$\begin{aligned} T(\dot{x}_1, \dot{x}_2, x_1, x_2) =& \frac{1}{2}m\left[\dot{x}_1^2 + \dot{x}_2^2 + \left(\frac{\partial S(x_1, x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial S(x_1, x_2)}{\partial x_2}\dot{x}_2\right)^2\right] \\ =& \frac{1}{2}m\left[\dot{x}_1^2\left(1 + \frac{\partial S(x_1, x_2)}{\partial x_1}\right) + \dot{x}_2^2\left(1 + \frac{\partial S(x_1, x_2)}{\partial x_2}\right) + 2\dot{x}_1\dot{x}_2\frac{\partial S(x_1, x_2)}{\partial x_1}\frac{\partial S(x_1, x_2)}{\partial x_1}\right]. \end{aligned} \tag{31}$$

On the other hand, the potential energy due to gravity is

$$V(x_1, x_2) = mgS(x_1, x_2). \tag{32}$$

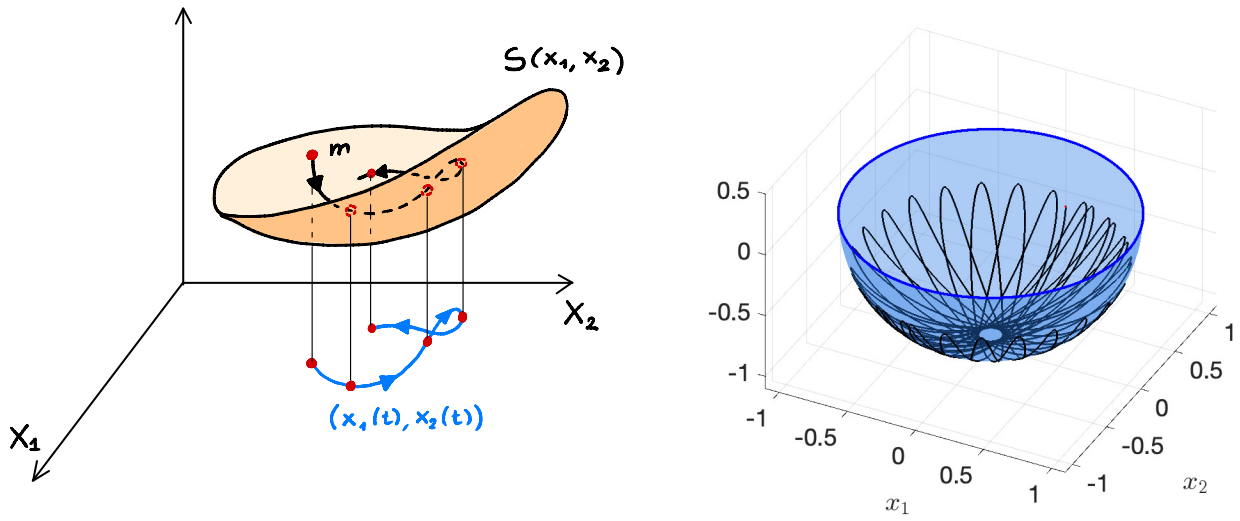


Figure 4: Dynamics of a point mass on a surface $S(x_1, x_2)$ under the effect of the gravity.

The total energy of the system is

$$E(\dot{x}_1, \dot{x}_2, x_1, x_2) = T(\dot{x}_1, \dot{x}_2, x_1, x_2) + V(x_1, x_2), \tag{33}$$

where T and V are given in (31) and (32). Since there is no friction $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$ is conserved along trajectories, i.e., the system is *conservative*. Hence, the trajectories are level sets of $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$. In other words, the relation between the coordinates (x_1, x_2) and velocity $(\dot{x}_1, \dot{x}_2)$ of the point mass m is fully determined by the level sets of $E(\dot{x}_1, \dot{x}_2, x_1, x_2)$. The value of the level set depends on the initial condition of the particle (position and velocity).

In the case where the surface S is a sphere, this system is called *spherical pendulum*. In the next course note on Lagrangian and Hamiltonian dynamics we will see how to derive the equations of motion for this system.

Double pendulum. Consider the double pendulum sketched in Figure 5. We can can express the position of the point masses m_1 and m_2 relative to the coordinate system (x, y) as

$$\begin{cases} x_1 = L_1 \sin(\theta_1) \\ y_1 = L_1 \cos(\theta_1) \end{cases} \qquad \begin{cases} x_2 = x_1 + L_2 \sin(\theta_2) \\ y_2 = y_1 + L_2 \cos(\theta_1) \end{cases} \tag{34}$$

The total potential energy is the sum of the potential gravitational energies of both masses

$$V(\theta_1, \theta_2) = (m_1 + m_2) g L_1 \left(1 - \cos(\theta_1)\right) + m_2 g L_2 \left(1 - \cos(\theta_2)\right). \tag{35}$$

On the other hand, the kinetic energy is

$$\begin{aligned} T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) =& \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ =& \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1^2 \dot{\theta}_2^2 \cos(\theta_1 - \theta_2)\right). \end{aligned} \tag{36}$$

Note that the kinetic energy is a quadratic form in $\dot{\theta}_1$ and $\dot{\theta}_2$. In this case, the total energy of the system

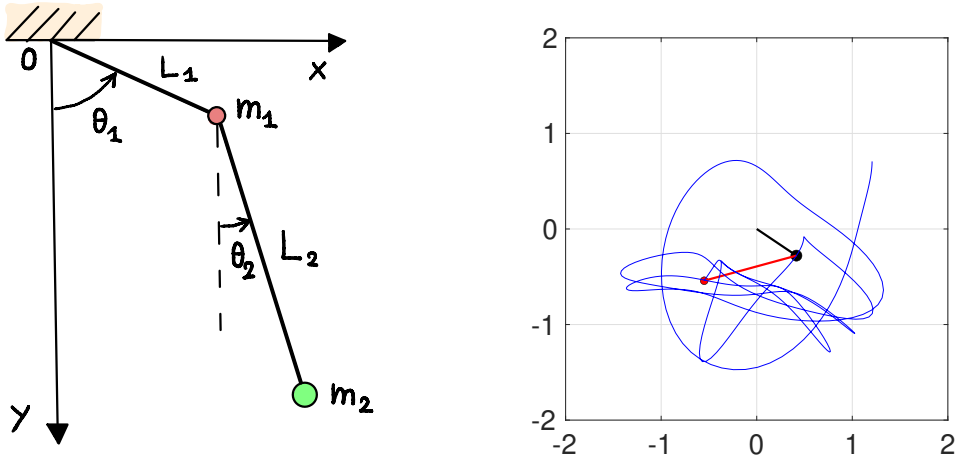


Figure 5: Sketch of a double pendulum. Gravity acts on both masses. There is also an interaction between m_1 and m_2 through the rod of length L_2.

can be written as

$$\begin{aligned} E(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) =& T(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) + V(\theta_1,\theta_2) \\ =& \frac{1}{2}m_1L_1^2\dot\theta_1^2 + \frac{1}{2}m_2\left(L_1^2\dot\theta_1^2 + L_2^2\dot\theta_2^2 + 2L_1L_2\dot\theta_1^2\dot\theta_2^2\cos(\theta_1-\theta_2)\right) + \\ & (m_1+m_2)gL_1\left(1-\cos(\theta_1)\right) + m_2gL_2\left(1-\cos(\theta_2)\right). \end{aligned} \tag{37}$$

Remarkably, the trajectories of the double pendulum can be chaotic. Yet, they are level sets of the four-dimensional energy function (37). In the next course note on Lagrangian and Hamiltonian dynamics we will see how to derive the equations of motion for this system.

Properties of conservative systems. Trajectories of conservative systems are level sets of some energy function $E(\boldsymbol{x})$ (see Equation (2)). Based on this observation it is straightforward to prove the following properties

- Relative maxima and relative minima of the energy function $E(\boldsymbol{x})$ are fixed points of the system. Moreover, trajectories nearby such fixed points are necessarily closed. In two dimensions maxima and minima of the energy function $E(\boldsymbol{x})$ are *centers* (see Figure 1 and Figure 3). In higher dimensions such local maxima and minima define a closed trajectory on a high-dimensional torus.

- Conservative systems cannot have stable/unstable nodes, stable/unstable spirals, limit cycles, and any other attracting or repelling set (e.g., fractal attractors) that contract/expand volume in its neighborhood. Note that conservative system can have lines of repelling or attracting fixed points though. For instance, the system

$$\begin{cases} \dot x_1 = -\sin(x_2)x_1 \\ \dot x_2 = \dfrac{2x_1^2x_2\sin(x_2)}{x_1^2+1} \end{cases} \tag{38}$$

  is conservative, with energy function $E(x_1,x_2) = x_1^2x_2 + x_2$. Trajectories are level sets of E but there can be lines of fixed points (zero velocity) that transverse such levels sets.

- Saddle nodes in 2D conservative system are saddle points of the energy function. This is clearly demonstrated in Figure 1 and Figure 3.

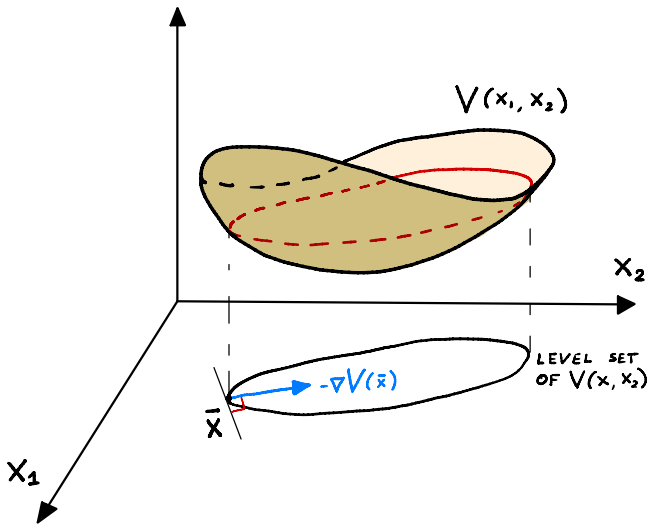


Figure 6: Geometric meaning of the gradient $-\nabla V(x_1, x_2)$.

Gradient systems. We say that the n-dimensional dynamical system (1) is a *gradient system* if it can be written in the form

$$\frac{d\boldsymbol{x}}{dt} = -\nabla V(\boldsymbol{x}), \tag{39}$$

for some scalar function $V(\boldsymbol{x})\colon \mathbb{R}^n \to \mathbb{R}$. In other words, a gradient system is a first-order dynamical system whose vector field $\boldsymbol{f}(\boldsymbol{x})$ is conservative[3] (see Appendix A for a review of conservative vector fields). The flow generated by a gradient system is called *gradient flow*.

The function $V(\boldsymbol{x})$ in (39) is called the *potential energy* associated with the vector field $\boldsymbol{f}(\boldsymbol{x})$. Note that $V(\boldsymbol{x})$ differs by a sign from the potential function $\varphi(\boldsymbol{x})$ discussed in Appendix A. To determine whether a given dynamical system is a gradient system, it is sufficient to check whether the vector field $\boldsymbol{f}(\boldsymbol{x})$ is conservative, i.e., whether it satisfies the conditions for path-integral independence (see Appendix A).

As shown in Figure 6, the vector $-\nabla V(\boldsymbol{x})$ is orthogonal to the level set of $V(\boldsymbol{x})$ passing through the point $\boldsymbol{x}$, and it points in the direction of steepest descent, i.e., the direction in which $V(\boldsymbol{x})$ decreases most rapidly. To prove that $\nabla V(\boldsymbol{x})$ is orthogonal to the level set $V(\boldsymbol{x}) = c$, let us parameterize such level set locally near the point $\boldsymbol{x}$ using a smooth curve $\boldsymbol{s}(\lambda)$ such that $\boldsymbol{s}(0) = \boldsymbol{x}$, where $\lambda \in \mathbb{R}$ is the parameter of the curve. Expanding $\boldsymbol{s}(\lambda)$ in a Taylor series about $\lambda = 0$ gives

$$\boldsymbol{s}(\lambda) = \boldsymbol{s}(0) + \frac{d\boldsymbol{s}}{d\lambda}(0)\,\lambda + \cdots . \tag{40}$$

Evaluating $V(\boldsymbol{x})$ along the curve $\boldsymbol{s}(\lambda)$ yields

$$V(\boldsymbol{s}(\lambda)) = V(\boldsymbol{s}(0)) + \nabla V(\boldsymbol{s}(0)) \cdot \frac{d\boldsymbol{s}}{d\lambda}(0)\,\lambda + \cdots . \tag{41}$$

Since the curve $\boldsymbol{s}(\lambda)$ lies entirely within the level set $V(\boldsymbol{x}) = c$, the function $V(\boldsymbol{s}(\lambda))$ is constant for all λ, and thus its derivative at $\lambda = 0$ must vanish

$$\frac{d}{d\lambda} V(\boldsymbol{s}(\lambda))\bigg|_{\lambda=0} = \nabla V(\boldsymbol{x}) \cdot \frac{d\boldsymbol{s}}{d\lambda}(0) = 0.$$

[3]It is important to distinguish second-order dynamical systems with potential forces (e.g., (25)) from first-order gradient systems of the form (39). Gradient systems are, in general, *not* conservative dynamical systems.

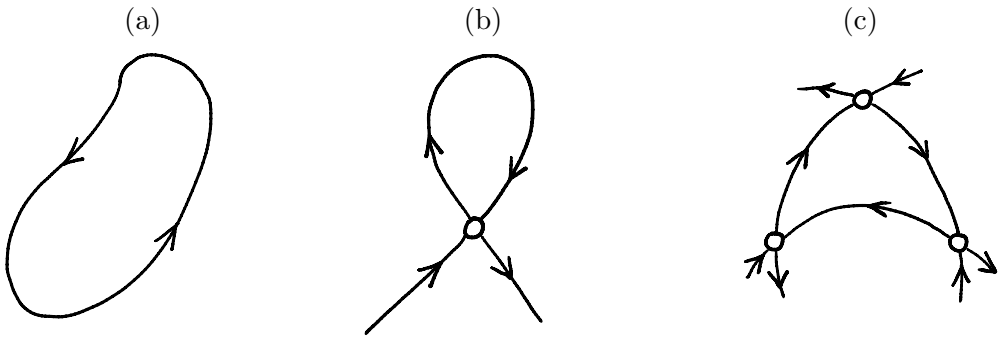


Figure 7: Gradient systems cannot have limit cycles (a), homoclinic orbits (b), or heteroclinic cycles (c).

This shows that the gradient $\nabla V(\boldsymbol{x})$ is orthogonal to the level set of $V(\boldsymbol{x})$ passing through $\boldsymbol{x}$. Indeed, $\dot{\boldsymbol{s}}(0)$ is the tangent vector to the curve $\boldsymbol{s}(\lambda)$ at the point $\boldsymbol{s}(0) = \boldsymbol{x}$, and therefore tangent to the level set at $\boldsymbol{x}$. Moreover, in gradient systems of the form $\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$, the trajectories follow the direction of steepest descent of V, and hence are orthogonal to its level sets[4].

Remark: Gradient systems play a pivotal role in *numerical optimization*, e.g., when computing the (local) minimum of a scalar function $V(\boldsymbol{x})$. In this context, the continuous-time gradient flow (39) represents a continuous analog of the well-known *gradient descent* algorithm, where the minimum of a cost function $V(\boldsymbol{x})$ is approximated iteratively via

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \gamma \nabla V(\boldsymbol{x}_k), \tag{42}$$

where $\gamma > 0$ is a real number referred to as the *learning rate*. Equation (42) can be thought of as an Euler forward scheme applied to the gradient system (39).

Clearly, the critical points of $V(\boldsymbol{x})$, i.e., maxima, minima, saddles, and any other point $\boldsymbol{x}^*$ satisfying

$$\nabla V(\boldsymbol{x}^*) = \boldsymbol{0}$$

are fixed points of the system (39). Moreover, $V(\boldsymbol{x})$ decreases monotonically along trajectories of the system (39) (recall that ∇V is orthogonal to the level sets of V). This can be also shown by a direct calculation,

$$\begin{aligned} V(\boldsymbol{x}(t_2)) =& V(\boldsymbol{x}(t_1)) + \int_{t_1}^{t_2} \frac{dV(\boldsymbol{x}(t))}{dt} dt \qquad t_2 \geq t_1 \\ =& V(\boldsymbol{x}(t_1)) + \int_{t_1}^{t_2} \nabla V(\boldsymbol{x}(t)) \cdot \frac{d\boldsymbol{x}(s)}{dt} dt \\ =& V(\boldsymbol{x}(t_1)) - \int_{t_1}^{t_2} \nabla V(\boldsymbol{x}(t)) \cdot \nabla V(\boldsymbol{x}(t)) dt \\ =& V(\boldsymbol{x}(t_1)) - \int_{t_1}^{t_2} \|\nabla V(\boldsymbol{x}(t))\|_2^2 \, dt \\ \leq& V(\boldsymbol{x}(t_1)), \end{aligned} \tag{43}$$

where the equality sign holds if and only if $\|\nabla V(\boldsymbol{x}(t))\|_2 = 0$, i.e., only at fixed points of (39). Equation (43) rules out the possibility of any closed orbit in gradient systems, including limit cycles, homoclinic orbits, and heteroclinic cycles (see Figure 7).

The Jacobian matrix of a gradient system is necessarily *symmetric* (see Eq. (45)). Therefore it can only have *real eigenvalues*. This implies that the flow of a gradient system nearby a hyperbolic fixed points cannot have any rotating component due to complex conjugate eigenvalues, i.e., no spirals. In other words, hyperbolic fixed points of gradient systems can only be sources, sinks, or saddle nodes!

[4] In contrast, for conservative systems, trajectories are themselves confined to level sets of an energy function.

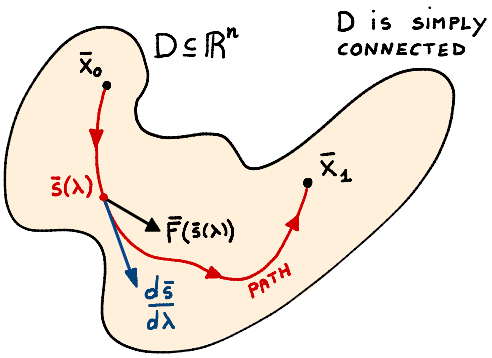


Figure 8: Sketch of a simply connected domain $D \subseteq \mathbb{R}^n$. Shown are two points $\boldsymbol{x}_0, \boldsymbol{x}_1 \in D$, a path $\boldsymbol{s}(\lambda)$ connecting $\boldsymbol{x}_0 = \boldsymbol{s}(0)$ to $\boldsymbol{x}_1 = \boldsymbol{s}(1)$ the vector field $\boldsymbol{F}(\boldsymbol{x})$ along the path, and the velocity vector $d\boldsymbol{s}(\lambda)/d\lambda$. If $\boldsymbol{F}(\boldsymbol{x})$ is conservative, then the line integral (46) does not depend on the path connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$.

Appendix A: Conservative vector fields

Let $\boldsymbol{F}(\boldsymbol{x})$ be a continuously differentiable vector field defined in a *simply connected* domain[5] $D \subseteq \mathbb{R}^n$. We say that $\boldsymbol{F}(\boldsymbol{x})$ is conservative if it can be written as a gradient of some function $\varphi : D \subseteq \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\boldsymbol{F}(\boldsymbol{x}) = \nabla \varphi(\boldsymbol{x}). \tag{44}$$

There are several equivalent conditions that characterize conservative vector fields in a simply connected domain:

a) The Jacobian of $\boldsymbol{F}(\boldsymbol{x})$ is symmetric, i.e.,

$$\frac{\partial F_i(\boldsymbol{x})}{\partial x_j} = \frac{\partial F_j(\boldsymbol{x})}{\partial x_i} \qquad \text{for all } \boldsymbol{x} \in D \text{ and for all } i, j = 1, \ldots, n. \tag{45}$$

In two and three dimensions this condition is equivalent to the well-known condition expressing the fact that $\boldsymbol{F}(\boldsymbol{x})$ is "irrotational" or curl-free.

b) The line integral of $\boldsymbol{F}(\boldsymbol{x})$ along a smooth curve $\boldsymbol{s}(\lambda)$ in D connecting two arbitrary points $\boldsymbol{x}_0, \boldsymbol{x}_1 \in D$

$$\varphi(\boldsymbol{x}_1) = \varphi(\boldsymbol{x}_0) + \int_0^1 \boldsymbol{F}(\boldsymbol{s}(\lambda)) \cdot \frac{d\boldsymbol{s}(\lambda)}{d\lambda} d\lambda \tag{46}$$

does not depend on the path connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$ (see Figure 8).

c) The line integral of $\boldsymbol{F}(\boldsymbol{x})$ along any smooth closed curve[6] $\boldsymbol{s}(\lambda)$ is zero, i.e.,

$$\int_0^1 \boldsymbol{F}(\boldsymbol{s}(\lambda)) \cdot \frac{d\boldsymbol{s}(\lambda)}{d\lambda} d\lambda = 0 \tag{47}$$

This follows immediately from (46) if we set $\boldsymbol{x}_1 = \boldsymbol{x}_0$.

It can be shown that (44), (45), and path independence of (46) all imply each other. For example, if the symmetry condition (45) is satisfied for all $\boldsymbol{x} \in D$ then there exists a potential $\varphi(\boldsymbol{x})$ satisfying (44).

[5] A domain $D \subseteq \mathbb{R}^n$ is called simply connected if any closed curve in D can be shrunk to a point without getting out of D.
[6] A closed curve satisfies $\boldsymbol{x}_0 = \boldsymbol{s}(0) = \boldsymbol{s}(1) = \boldsymbol{x}_1$.

The potential $\varphi(\boldsymbol{x})$ can be determined by computing the line integral (46) along *any* path connecting two points in D. In particular, if D is convex we can consider a simple line as a path

$$\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{x}_0 + \lambda \boldsymbol{x}_1 \qquad \lambda \in [0,1]. \tag{48}$$

Example: Consider the two dimensional vector field $\boldsymbol{F}(\boldsymbol{x}) = (F_1(x_1,x_2), F_2(x_1,x_2))$ where

$$F_1(x_1,x_2) = 2x_1x_2, \qquad F_2(x_1,x_2) = x_1^2. \tag{49}$$

It is straightforward to show that $\boldsymbol{F}(\boldsymbol{x})$ conservative. In fact, the Jacobain of $\boldsymbol{F}$ is symmetric for all $\boldsymbol{x} \in \mathbb{R}^2$

$$\frac{\partial F_1(x_1,x_2)}{\partial x_2} = 2x_1 = \frac{\partial F_2(x_1,x_2)}{\partial x_1}. \tag{50}$$

Therefore the symmetry condition (45) is satisfied. To compute a potential $\varphi(\boldsymbol{x})$, we are free to choose any curve we like to evaluate the integral (46). So here we consider a line (48) connecting an arbitrary point $\boldsymbol{x} \in \mathbb{R}^2$ to the origin, i.e.,

$$\boldsymbol{s}(\lambda) = \lambda \boldsymbol{x} \qquad \frac{d\boldsymbol{s}(\lambda)}{d\lambda} = \boldsymbol{x}. \tag{51}$$

We have,

$$\begin{aligned}
\varphi(\boldsymbol{x}) =& \varphi(\boldsymbol{0}) + \int_0^1 \boldsymbol{F}(\boldsymbol{s}(\lambda)) \cdot \frac{d\boldsymbol{s}(\lambda)}{d\lambda} d\lambda \\
=& \varphi(\boldsymbol{0}) + \int_0^1 \boldsymbol{F}(\lambda \boldsymbol{x}) \cdot \boldsymbol{x} d\lambda \\
=& \varphi(\boldsymbol{0}) + \int_0^1 \left(2\lambda^2 x_1^2 x_2 + \lambda^2 x_1^2 x_2\right) d\lambda \\
=& \varphi(\boldsymbol{0}) + x_1^2 x_2.
\end{aligned} \tag{52}$$

The additive constant $\varphi(\boldsymbol{0})$ does not change the gradient representing $\boldsymbol{F}(\boldsymbol{x})$ in (44). Therefore we can set it to zero. This yields the potential

$$\varphi(\boldsymbol{x}) = x_1^2 x_2. \tag{53}$$

Let us verify that the gradient of (53) coincides with $\boldsymbol{F}(\boldsymbol{x})$. We have

$$\varphi(\boldsymbol{x}) = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2} \right). \tag{54}$$

yields the vector field (49). We have,

$$\begin{aligned}
\frac{\partial \varphi}{\partial x_1} =& 2x_1x_2 \qquad \text{(same as } F_1(x_1,x_2)\text{)}, \\
\frac{\partial \varphi}{\partial x_2} =& x_1^2 \qquad \text{(same as } F_2(x_1,x_2)\text{)}.
\end{aligned}$$

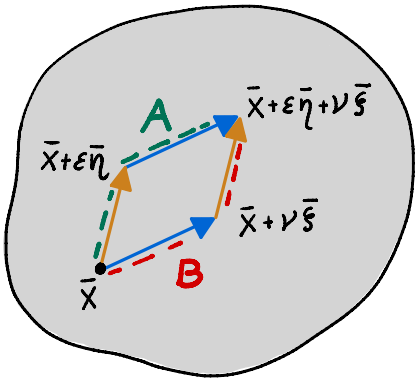
Path invariance implies symmetry of the Jacobian. It is useful to show how (45) follows directly from (46). To this end, consider a point $\boldsymbol{x}$ and its infinitesimal neighborhood. All points in such neighborhood can be connected via two straight lines as follows

By imposing that the line integral of $\boldsymbol{F}(\boldsymbol{x})$ along the paths A and B above are identical we obtain[7]

$$\epsilon \boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{\eta} + \nu \boldsymbol{F}(\boldsymbol{x} + \epsilon \boldsymbol{\eta}) \cdot \boldsymbol{\xi} = \nu \boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{\xi} + \epsilon \boldsymbol{F}(\boldsymbol{x} + \nu \boldsymbol{\xi}) \cdot \boldsymbol{\xi}. \tag{57}$$

[7]Note that along the infinitesimal path (line) $\boldsymbol{s}(\epsilon) = \boldsymbol{x} + \epsilon \boldsymbol{\eta}$

$$\int_0^\epsilon \boldsymbol{F}(\boldsymbol{s}(\epsilon)) \cdot \frac{d\boldsymbol{s}(\epsilon)}{d\epsilon} d\epsilon = \epsilon \boldsymbol{F}(\boldsymbol{x}) \cdot \boldsymbol{\eta}. \tag{55}$$



This equation can be simplified further by expanding $\boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{\eta})$ and $\boldsymbol{F}(\boldsymbol{x}+\nu\boldsymbol{\xi})$ in a first-order Taylor series, i.e.,

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{\eta}) \simeq &\boldsymbol{F}(\boldsymbol{x})+\epsilon\boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{x})\boldsymbol{\eta},\\ \boldsymbol{F}(\boldsymbol{x}+\nu\boldsymbol{\xi}) \simeq &\boldsymbol{F}(\boldsymbol{x})+\epsilon\boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{x})\boldsymbol{\nu}.\end{aligned} \tag{58}$$

A substitution of these expansions into (57) yields

$$[\boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{x})\boldsymbol{\eta}]\cdot\boldsymbol{\xi}=[\boldsymbol{J}_{\boldsymbol{F}}(\boldsymbol{x})\boldsymbol{\xi}]\cdot\boldsymbol{\eta}. \tag{59}$$

This condition implies (for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ arbitrary) that the Jacobian of $\boldsymbol{F}(\boldsymbol{x})$ must be a symmetric matrix at $\boldsymbol{x}$.

Similarly, along the path $\boldsymbol{s}(\nu)=\boldsymbol{x}+\epsilon\boldsymbol{\eta}+\nu\boldsymbol{\xi}$ connecting $\boldsymbol{x}+\epsilon\boldsymbol{\eta}$ to $\boldsymbol{x}+\epsilon\boldsymbol{\eta}+\nu\boldsymbol{\xi}$ we have, for small ν,

$$\int_0^{\nu}\boldsymbol{F}(\boldsymbol{s}(\nu))\cdot\frac{d\boldsymbol{s}(\nu)}{d\nu}d\nu=\nu\boldsymbol{F}(\boldsymbol{x}+\epsilon\boldsymbol{\eta})\cdot\boldsymbol{\xi}. \tag{56}$$