Lagrangian and Hamiltonian dynamics

In this course note we provide a brief introduction to Lagrangian and Hamiltonian dynamics, and show some applications. To this end, we consider a system with configuration described by n *generalized coordinates*

$$\boldsymbol{q}(t) = (q_1(t), q_2(t), \ldots, q_n(t)), \tag{1}$$

and corresponding *generalized velocities*

$$\dot{\boldsymbol{q}}(t) = (\dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t)). \tag{2}$$

The generalized coordinates $\boldsymbol{q}(t)$ can be any set of independent variables, e.g., angles and displacements, that uniquely identify the configuration of the system. The state system is $2n$-dimensional and defined by the pair $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Kinetic energy, potential energy, and total energy. We consider dynamical systems in which the kinetic energy can be written as

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{q}, t)\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} b_i(\boldsymbol{q}, t)\dot{q}_i + c(\boldsymbol{q}, t), \tag{3}$$

where $a_{ij}(\boldsymbol{q}, t)$ is a symmetric, positive-definite matrix[1] for all $\boldsymbol{q}$ and t. We will see examples of systems with kinetic in the form (3) later in this note. Furthermore we assume that the potential energy $V(\boldsymbol{q}, t)$ depends only on the configuration variables $\boldsymbol{q}$ and eventually time. The *total energy* of the system can be written as

$$E(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + V(\boldsymbol{q}, t). \tag{4}$$

Note that due to the time dependence, the total energy $E(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ may not be conserved in time, i.e., the system may not be conservative. On the other hand, if $E(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ does not depend on time then E is a conserved quantity.

Lagrangian function and the principle of stationary action. Define the *Lagrangian function*

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - V(\boldsymbol{q}, t), \tag{5}$$

and the *action functional*

$$\mathcal{A}([\boldsymbol{q}]) = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)\, dt. \tag{6}$$

The Lagrangian function contains all physical information concerning the system and the forces acting on it. Remarkably, the differential equations governing the dynamics of the system can be derived based on the knowledge of the Lagrangian (5) and the following principle.

- ***Principle of stationary action (Hamilton's principle):*** The dynamics of a nonlinear dynamical system with Lagrangian function (5) makes the action (6) stationary relative to all possible paths $\boldsymbol{q}(t)$ connecting two given admissible configurations $\boldsymbol{q}(t_1)$ to $\boldsymbol{q}(t_2)$.

The basic idea of Hamilton's principle is sketched in Figure 1, and it can be expressed mathematically as

$$\delta\mathcal{A}([\boldsymbol{q}]) = \mathcal{A}([\boldsymbol{q} + \epsilon\boldsymbol{\eta}]) - \mathcal{A}([\boldsymbol{q}]) = 0, \tag{7}$$

for arbitrary perturbations $\boldsymbol{\eta}(t)$ satisfying

$$\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = 0. \tag{8}$$

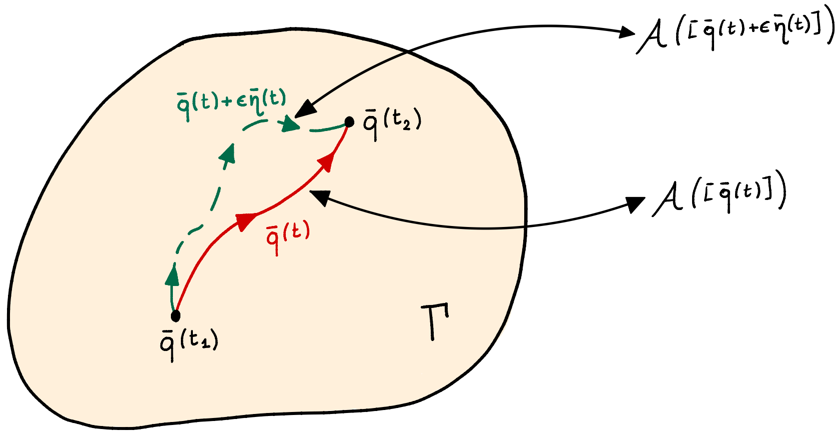

Figure 1: Sketch of the configuration space Γ and two paths connecting two admissible configurations $\boldsymbol{q}(t_0)$ and $\boldsymbol{q}(t_1)$. The path taken by the system makes the action (6) stationary (i.e., $\delta\mathcal{A}([\boldsymbol{q}]) = 0$) relative to arbitrary small perturbations $\epsilon\boldsymbol{\eta}(t)$ satisfying $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}$.

The quantity $\delta\mathcal{A}$ is the called first variation of the functional $\mathcal{A}([\boldsymbol{q}])$ at $\boldsymbol{q}(t)$.

Euler-Lagrange equations. Using elementary calculus of variations applied to the action functional (6), it is possible to derive the equations of motion of a nonlinear system just based on the specification of the Lagrangian function (5). This makes the process of deriving such equations very straightforward, and it allows us to proceed in those cases where physical intuition may be difficult to apply. To this end, consider perturbed trajectory

$$\widetilde{\boldsymbol{q}}(t) = \boldsymbol{q}(t) + \epsilon\boldsymbol{\eta}(t) \tag{9}$$

where $\boldsymbol{\eta}(t)$ is an arbitrary function satisfying the conditions $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = \mathbf{0}$, so that each path $\widetilde{\boldsymbol{q}}(t)$ has the same endpoints. By evaluating the action functional (6) at $\boldsymbol{q}(t) + \epsilon\boldsymbol{\eta}(t)$ we obtain

$$\mathcal{A}([\boldsymbol{q} + \epsilon\boldsymbol{\eta}]) = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}(t) + \epsilon\boldsymbol{\eta}(t), \dot{\boldsymbol{q}}(t) + \epsilon\dot{\boldsymbol{\eta}}(t), t)dt. \tag{10}$$

The Hamilton's principle can be formulated mathematically as[2]

$$\left.\frac{d\mathcal{A}[\boldsymbol{q} + \epsilon\boldsymbol{\eta}]}{d\epsilon}\right|_{\epsilon=0} = \frac{d}{d\epsilon}\left[\int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{q}(t) + \epsilon\boldsymbol{\eta}(t), \dot{\boldsymbol{q}}(t) + \epsilon\dot{\boldsymbol{\eta}}(t), t)dt\right]_{\epsilon=0} = 0. \tag{12}$$

Developing the derivatives and evaluating everything at $\epsilon = 0$ yields

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{\partial\mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial q_i}\eta_i(t) + \frac{\partial\mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial \dot{q}_i}\dot{\eta}_i(t)\right] dt = 0. \tag{13}$$

By using integration by parts and recalling that the boundary conditions for $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_1) = \mathbf{0}$ are zero we can write

$$\int_{t_1}^{t_2} \frac{\partial\mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial \dot{q}_i}\dot{\eta}_i(t)dt = \underbrace{\left[\frac{\partial\mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial \dot{q}_i}\eta_i(t)\right]_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial\mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial \dot{q}_i}\eta_i(t)dt. \tag{14}$$

[1]Recall that positive-definite matrices have strictly positive eigenvalues and therefore they are necessarily invertible.

[2]Note that to first order in ϵ we have

$$\delta\mathcal{A}([\boldsymbol{q}]) = \epsilon \left.\frac{d\mathcal{A}[\boldsymbol{q} + \epsilon\boldsymbol{\eta}]}{d\epsilon}\right|_{\epsilon=0}. \tag{11}$$

Hence (12) is equivalent to (7).

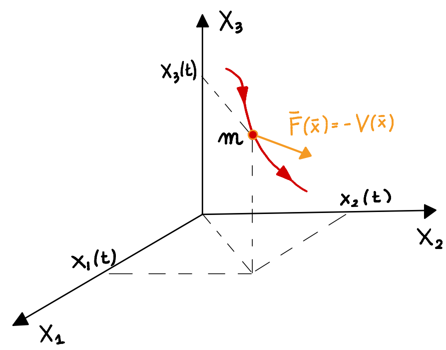

Figure 2: Dynamics of a point mass in 3D under the action of a conservative vector field $\boldsymbol{F}(\boldsymbol{x})$.

Substituting this back into (13) yields

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)}{\partial \dot{q}_i} \right] \eta_i(t) dt = 0. \tag{15}$$

Finally, recalling that $\eta_i(t)$ is arbitrary we obtain the *Euler-Lagrange equations*

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad i = 1, \ldots, n.} \tag{16}$$

This is a system of n second-order nonlinear ODEs. The system is not necessarily in a normal form[3].

Dynamics of a point mass in three-dimensional space. Consider the dynamics of a point mass m subject to conservative vector field $\boldsymbol{F}(\boldsymbol{x}) = -\nabla V(\boldsymbol{x})$. Let

$$\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t)), \tag{18}$$

be the coordinates of the particle relative to a Cartesian coordinate system (see Figure 2). Such coordinates identify the configuration of the system, i.e., they are the variables $\boldsymbol{q}(t)$ of this problem. The Lagrangian for this system is

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \underbrace{\frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right)}_{\text{kinetic energy}} - V(\boldsymbol{x}) \tag{19}$$

Differentiating the Lagrangian with respect to x_i and $\dot{x}_i$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i, \qquad \frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial V}{\partial x_i} \quad i = 1, 2, 3 \tag{20}$$

and substituting these expressions into the Euler-Lagrange equations (16) yields

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i} \quad \Rightarrow \quad m\frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i} = F_i(\boldsymbol{x}). \tag{21}$$

Hence, the Hamilton's principle and the corresponding Euler-Lagrange equations for the Lagrangian (19) are completely equivalent to the Netwton's equations of motion in this case.

[3] A system of second-order differential equations is said to be in a *normal form* if the second derivatives can be expressed explicitly as

$$\ddot{\boldsymbol{x}} = \boldsymbol{g}(\dot{\boldsymbol{x}}, \boldsymbol{x}, t). \tag{17}$$

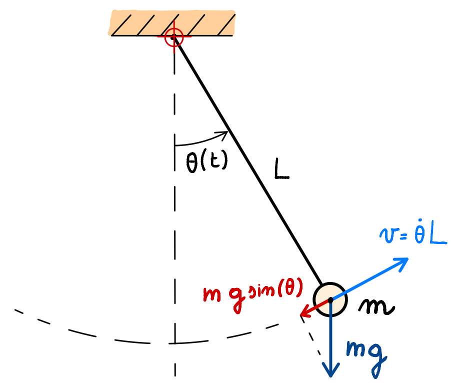

Figure 3: Sketch of a simple pendulum. The pendulum is assumed to have no friction, i.e., the only external force acting on the point mass m is gravity.

Nonlinear pendulum. Consider the pendulum sketched in Figure 3. The configuration variable is $q(t) = \theta(t)$, while the state of the system is specified by $(q, \dot{q}) = (\theta, \dot{\theta})$. The kinetic energy of the pendulum is

$$T(\dot{\theta}) = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\dot{\theta}^2 \tag{22}$$

while the potential energy relative to the vertical position is

$$V(\theta) = mgL\left(1 - \cos(\theta)\right) = -mgL\cos(\theta) + C, \tag{23}$$

where C is a constant. Hence, the Lagrangian for the pendulum is

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}mL^2\dot{\theta}^2 - mgL\left(1 - \cos(\theta)\right). \tag{24}$$

Differentiating the Lagrangian with respect to $\dot{\theta}$ and θ

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta}, \qquad \frac{\partial \mathcal{L}}{\partial \theta} = -mgL\sin(\theta) \tag{25}$$

and substituting the derivatives into the Euler-Lagrange equations (16) yields

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{L}\sin(\theta), \tag{26}$$

i.e., we obtain the differential equation of motion governing the dynamics of the pendulum.

Double pendulum. Consider the double pendulum sketched in Figure 4 We choose the configuration variables to be $(\theta_1(t), \theta_2(t))$. We have seen in the course note 6 that the potential energy and the kinetic potential energy for this system can be written as

$$V(\theta_1, \theta_2) = -(m_1 + m_2)\cos(\theta_1) - m_2 g L_2 \cos(\theta_2). \tag{27}$$

On the other hand, the kinetic energy is

$$T(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}\left(m_1 + m_2\right)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 L_2^2\dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2). \tag{28}$$

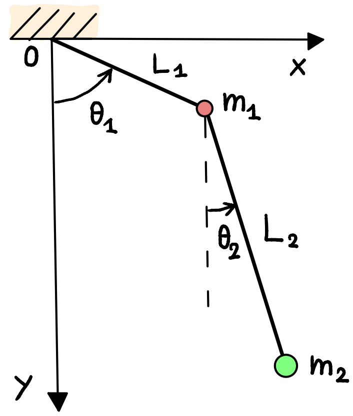

Figure 4: Sketch of a double pendulum. Gravity acts on both masses. There is also an interaction between m_1 and m_2 through the rod of length L_2.

Hence, the Lagrangian function for this system is

$$\begin{aligned}\mathcal{L}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) =& \frac{1}{2}\left(m_1+m_2\right)L_1^2\dot\theta_1^2+\frac{1}{2}m_2L_2^2\dot\theta_2^2+m_2L_1L_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)+\\ &(m_1+m_2)\cos(\theta_1)+m_2gL_2\cos(\theta_2),\end{aligned} \tag{29}$$

while the Euler Lagrange equations (16) take the form

$$\begin{cases}\dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot\theta_1}\right)=\dfrac{\partial\mathcal{L}}{\partial\theta_1},\\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot\theta_2}\right)=\dfrac{\partial\mathcal{L}}{\partial\theta_2}.\end{cases} \tag{30}$$

A rather lengthy calculation of all derivatives in (30) yields the system

$$\ddot\theta_1+a_1(\theta_1,\theta_2)\ddot\theta_2 = b_1\left(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2\right), \tag{31}$$

$$\ddot\theta_2+a_2(\theta_1,\theta_2)\ddot\theta_1 = b_2\left(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2\right), \tag{32}$$

where

$$a_1(\theta_1,\theta_2) = \frac{L_2}{L_1}\left(\frac{m_2}{m_1+m_2}\right)\cos(\theta_1-\theta_2), \tag{33}$$

$$a_2(\theta_1,\theta_2) = \frac{L_1}{L_2}\cos(\theta_1-\theta_2), \tag{34}$$

and

$$b_1\left(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2\right) = -\frac{L_2}{L_1}\left(\frac{m_2}{m_1+m_2}\right)\dot\theta_1^2\sin(\theta_1-\theta_2)-\frac{g}{L_1}\sin(\theta_1), \tag{35}$$

$$b_2\left(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2\right) = \frac{L_1}{L_2}\dot\theta_1^2\sin(\theta_1-\theta_2)-\frac{g}{L_2}\sin(\theta_2). \tag{36}$$

Note that the system (31)-(32) can be written in a matrix-vector form as

$$\begin{bmatrix}1 & a_1(\theta_1,\theta_2)\\ a_2(\theta_1,\theta_2) & 1\end{bmatrix}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix}=\begin{bmatrix}b_1(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)\\ b_2(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)\end{bmatrix}. \tag{37}$$

Note that the system (37) is not in a normal form, but it can be transformed to a normal form by inverting the matrix multiplying $(\ddot{\theta}_1, \ddot{\theta}_2)$ at left hand side of (37), i.e.,

$$\begin{bmatrix} 1 & a_1(\theta_1,\theta_2) \\ a_2(\theta_1,\theta_2) & 1 \end{bmatrix}. \tag{38}$$

Such matrix is invertible since

$$\det\left(\begin{bmatrix} 1 & a_1(\theta_1,\theta_2) \\ a_2(\theta_1,\theta_2) & 1 \end{bmatrix}\right) = 1 - a_1(\theta_1,\theta_2)a_2(\theta_1,\theta_2) = 1 - \left(\frac{m_2}{m_1+m_2}\right)\cos^2(\theta_1-\theta_2) > 0. \tag{39}$$

Hamiltonian dynamics. Consider the Lagrangian function (5) and define the *canonical momenta*

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i = 1,\ldots,n. \tag{40}$$

A substitution of the Lagrangian (5) into (40) yields

$$p_i = \sum_{j=1}^{n} a_{ij}(\boldsymbol{q},t)\dot{\boldsymbol{q}}_j + b_i(\boldsymbol{q},t). \tag{41}$$

The matrix $a_{ij}(\boldsymbol{q},t)$ is, by hypothesis, symmetric and positive definite. Therefore it is always invertible. This implies that the linear system

$$\underbrace{\begin{bmatrix} a_{11}(\boldsymbol{q},t) & \cdots & a_{1n}(\boldsymbol{q},t) \\ \vdots & \ddots & \vdots \\ a_{n1}(\boldsymbol{q},t) & \cdots & a_{nn}(\boldsymbol{q},t) \end{bmatrix}}_{\boldsymbol{A}(\boldsymbol{q},t)} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}}_{\dot{\boldsymbol{q}}} = \underbrace{\begin{bmatrix} p_1 - b_1(\boldsymbol{q},t) \\ \vdots \\ p_n - b_n(\boldsymbol{q},t) \end{bmatrix}}_{\boldsymbol{p}-\boldsymbol{b}(\boldsymbol{q},t)} \tag{42}$$

can be uniquely solved for $\dot{\boldsymbol{q}}$. In other words, we can uniquely express the generalized velocities $\dot{\boldsymbol{q}}(t)$ as a function of the canonical momenta $\boldsymbol{p}(t)$, and vice versa. This transformation is called *Legendre transformation* and it is written explicitly as

$$\dot{\boldsymbol{q}} = \boldsymbol{A}^{-1}(\boldsymbol{q},t)\left[\boldsymbol{p} - \boldsymbol{b}(\boldsymbol{q},t)\right]. \tag{43}$$

Next we define the Hamilton's function (or Hamiltonian)

$$\mathcal{H}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \sum_{i=1}^{n} \underbrace{\frac{\partial \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial \dot{q}_i}}_{p_i(\boldsymbol{q},\dot{\boldsymbol{q}},t)} \dot{q}_i - \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}},t). \tag{44}$$

Clearly, by using the Legendre transformation (43) it is possible to write the Hamilton's function as a function of $\boldsymbol{p}$, $\boldsymbol{q}$, and t. Hereafter we provide two fundamental theorems of Hamiltonian dynamics.

Theorem 1 (Hamilton's equations)**.** Consider a nonlinear system with Lagrangian (5) and Hamiltonian (44). The dynamics of the system is governed by the following system of $2n$ first-order ordinary differential equations in a normal form

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}(\boldsymbol{p},\boldsymbol{q},t)}{\partial p_i}, \\[2ex] \dot{p}_i = -\dfrac{\partial \mathcal{H}(\boldsymbol{p},\boldsymbol{q},t)}{\partial q_i}. \end{cases} \tag{45}$$

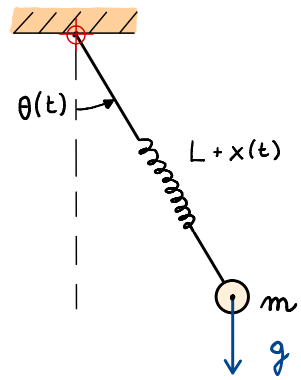

Figure 5: Sketch of an elastic pendulum. The spring can be compressed or extended and it generates a force $F(x) = -\kappa x$, where κ is the elastic constant of the spring.

Theorem 2 (Conservative systems)**.** If the Lagrangian (5) does not explicitly depend on time t, then the Hamiltonian (44) is conserved along trajectories of the system. In other words, every system described by a time-independent Lagrangian is conservative[4], and its trajectories are level sets of the Hamiltonian $\mathcal{H}(\boldsymbol{p}, \boldsymbol{q})$. Note that the Hamiltonian coincides with the total energy of the system.

The proofs of Theorem 1 and Theorem 2 are given in Appendix A. Note that Hamilton's equations are *always* in a normal form. This means that they are always in a form that can be easily integrated numerically.

Symplectic geometry. The geometric properties of vector fields associated with Hamiltonian dynamical systems are extremely interesting from a mathematical viewpoint. Indeed, they ended up generating a new research field in mathematics called *symplectic geometry*. Symplectic geometry has its origins in the Hamiltonian formulation of classical mechanics where the phase space of classical systems takes on the structure of a symplectic manifold. It can be shown that Hamiltonian dynamics (both conservative and non-conservative) is always *volume-preserving* in the phase space. This implies that Hamiltonian systems, whether they are conservative or not, cannot have attractors or repellors.

Hamiltonian formulation of the elastic pendulum. Consider the elastic pendulum sketched in Figure 5. The configuration space is two-dimensional and defined by the variables $x(t)$ and $\theta(t)$. Hence the phase space is four-dimensional. The kinetic and the potential energy for for this system are, respectively,

$$T = \frac{1}{2} m \left(\dot{x}^2 + (L+x)^2 \dot{\theta}^2\right), \tag{46}$$

$$V = -mg(L+x)\cos(\theta) + \frac{1}{2}\kappa x^2, \tag{47}$$

where κ is the spring elasticity constant. Hence the Lagrangian $\mathcal{L} = T - V$ is

$$\mathcal{L}(x, \theta, \dot{x}, \dot{\theta}) = \frac{1}{2} m \left(\dot{x}^2 + (L+x)^2 \dot{\theta}^2\right) + mg(L+x)\cos(\theta) - \frac{1}{2}\kappa x^2. \tag{48}$$

The canonical momenta (40) are

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \tag{49}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m(L+x)^2 \dot{\theta}. \tag{50}$$

[4]We proved this result using Nöether's theorem (Eq. 94), as well as a direct calculation (Appendix A).

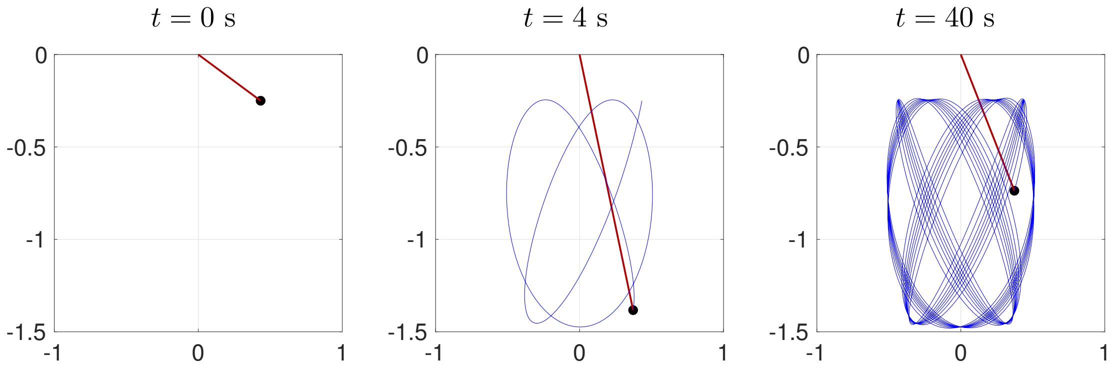

Figure 6: Numerical simulation of the elastic pendulum shown in Figure 5 with mass $m_1 = 3.5$ kg, length at rest $L = 0.5$ m, and spring constant $\kappa = 50$ N/m. The solution is computed by integrating the Hamilton's equations (53) numerically with RK4.

In this case the Legendre transformation (43) is trivial. The Hamiltonian (44) can be written as

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{x} + \frac{\partial \mathcal{L}}{\partial \dot{\theta}}\dot{\theta} - \mathcal{L}(x, \theta, \dot{x}, \dot{\theta}). \tag{51}$$

This can be written in terms of the canonical variables p_x, p_θ, x and θ as

$$\begin{aligned} \mathcal{H} =& \frac{p_x^2}{m} + \frac{p_\theta^2}{m(L+x)^2} - \frac{1}{2}\left(\frac{p_x^2}{m^2} + (L+x)^2\frac{p_\theta^2}{m^2(L+x)^4}\right) - mg(L+x)\cos(\theta)\frac{1}{2}\kappa x \\ =& \frac{p_x^2}{2m} + \frac{p_\theta^2}{2m(L+x)^2} - mg(L+x)\cos(\theta) + \frac{1}{2}\kappa x^2. \end{aligned} \tag{52}$$

This yields the following system Hamilton's equation

$$\begin{cases} \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p_x} = \dfrac{p_x}{m}, \\ \dot{\theta} = \dfrac{\partial \mathcal{H}}{\partial p_\theta} = \dfrac{p_\theta}{m(L+x)^2}, \\ \dot{p}_x = -\dfrac{\partial \mathcal{H}}{\partial x} = \dfrac{p_\theta^2}{m(L+x)^3} + mg\cos(\theta) - \kappa x, \\ \dot{p}_\theta = -\dfrac{\partial \mathcal{H}}{\partial \theta} = -mg(L+x)\sin(\theta). \end{cases} \tag{53}$$

The spring-mass pendulum can exhibit chaotic behavior. The nonlinear system of equations (53) is solved numerically in Figure 6. .

Hamiltonian formulation of the double pendulum. Consider the double pendulum sketched in Figure 4. The Lagrangian is given in (29) and it is hereafter rewritten for convenience

$$\begin{aligned} \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) =& \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \\ & (m_1 + m_2)\cos(\theta_1) + m_2 g L_2 \cos(\theta_2). \end{aligned} \tag{54}$$

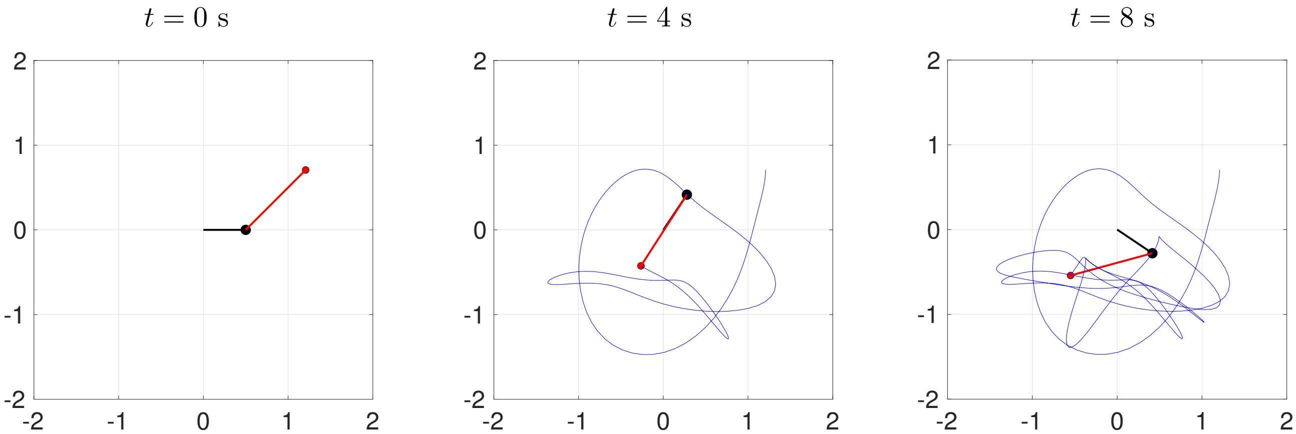

Figure 7: Numerical simulation of a double pendulum with masses $m_1 = 4$ kg, $m_2 = 2$ kg and lengths $L_1 = 1$ m, $L_2 = 0.5$ m. The solution is computed by integrating the Hamilton's equations (62)-(**??**) numerically with RK4.

The canonical momenta (40) are

$$p_{\theta_1} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 + m_2)L_1^2\dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2), \tag{55}$$

$$p_{\theta_2} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2). \tag{56}$$

This linear system can be written as

$$\underbrace{\begin{bmatrix} (m_1 + m_2)L_1^2 & m_2 L_1 L_2 \cos(\theta_2 - \theta_1) \\ m_2 L_1 L_2 \cos(\theta_2 - \theta_1) & m_2 L_2^2 \end{bmatrix}}_{\boldsymbol{A}(\theta_1,\theta_2)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} p_{\theta_1} \\ p_{\theta_2} \end{bmatrix}. \tag{57}$$

The matrix $\boldsymbol{A}(\theta_1, \theta_2)$ is symmetric and positive definite. Therefore it can be inverted to obtain (see Eq. (43))

$$\dot{\theta}_1 = \frac{L_2 p_{\theta_1} - L_1 p_{\theta_2} \cos(\theta_1 - \theta_2)}{L_2 L_1^2 \left(m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right)}, \tag{58}$$

$$\dot{\theta}_2 = \frac{-m_2 L_2 p_{\theta_1} \cos(\theta_1 - \theta_2) + (m_1 + m_2) L_1 p_{\theta_2}}{m_2 L_1 L_2^2 \left(m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right)}. \tag{59}$$

The Hamilton's function (44) for this system is

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\dot{\theta}_1 + \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\dot{\theta}_2 - \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_1). \tag{60}$$

A substitution of (54)-(56) and (58)-(59) into (60) yields

$$\begin{aligned}\mathcal{H}(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2) =& \frac{m_2 L_2^2 p_{\theta_1}^2 + (m_1 + m_2) L_1^2 p_{\theta_2}^2 - 2 m_2 L_1 L_2 p_{\theta_1} p_{\theta_2} \cos(\theta_1 - \theta_2)}{2 m_2 L_1^2 L_2^2 \left(m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right)} - \\ & (m_1 + m_2) g L_1 \cos(\theta_1) - m_2 g L_2 \cos(\theta_2). \end{aligned} \tag{61}$$

Hence, the Hamilton's equations of motion are

$$\begin{cases} \dot{\theta}_1 = \dfrac{\partial \mathcal{H}}{\partial p_{\theta_1}} = \dfrac{L_2 p_{\theta_1} - L_1 p_{\theta_2}\cos(\theta_1-\theta_2)}{L_2 L_1^2\left(m_1 + m_2\sin^2(\theta_1-\theta_2)\right)}, \\ \dot{\theta}_2 = \dfrac{\partial \mathcal{H}}{\partial p_{\theta_2}} = \dfrac{-m_2 L_2 p_{\theta_1}\cos(\theta_1-\theta_2) + (m_1+m_2)L_1 p_{\theta_2}}{m_2 L_1 L_2^2\left(m_1 + m_2\sin^2(\theta_1-\theta_2)\right)}, \\ \dot{p}_{\theta_1} = -\dfrac{\partial \mathcal{H}}{\partial \theta_1} = -(m_1+m_2)gL_1\sin(\theta_1) - \alpha(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2), \\ \dot{p}_{\theta_2} = -\dfrac{\partial \mathcal{H}}{\partial \theta_2} = -m_2 g L_2 \sin(\theta_1) + \alpha(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2), \end{cases} \tag{62}$$

where

$$\begin{aligned} \alpha(p_{\theta_1}, p_{\theta_2}, \theta_1, \theta_2) =& \frac{p_{\theta_1} p_{\theta_2}\sin(\theta_1-\theta_2)}{L_1 L_2(m_1 + m_2\sin^2(\theta_1-\theta_2))} - \\ & \sin(2(\theta_1-\theta_2))\frac{m_2 L_2^2 p_{\theta_1}^2 + (m_1+m_2)L_1^1 p_{\theta_2}^2 - 2m_1 L_1 L_2 p_{\theta_1} p_{\theta_2}\cos(\theta_1-\theta_2)}{2L_1^2 L_2^2(m_1 + m_2\sin^2(\theta_1-\theta_2))}. \end{aligned} \tag{63}$$

The pendulum system can exhibit chaotic behavior. A numerical solution of Hamilton's equations (62) is shown in Figure 6.

Two-dimensional Hamiltonian systems. A two-dimensional Hamiltonian system can be written in the form

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{\partial \mathcal{H}(x_1,x_2)}{\partial x_2} \\ \dfrac{dx_1}{dt} = -\dfrac{\partial \mathcal{H}(x_1,x_2)}{\partial x_1} \end{cases} \tag{64}$$

Hence, given a vector field $\boldsymbol{f}(x_1, x_2)$, if we can find a function $\mathcal{H}(x_1, x_2)$ such that

$$f_1(x_1,x_2) = \frac{\partial \mathcal{H}(x_1,x_2)}{\partial x_2} \qquad f_2(x_1,x_2) = -\frac{\partial \mathcal{H}(x_1,x_2)}{\partial x_1} \tag{65}$$

then we can immediately conclude that the system is conservative. In fact,

$$\begin{aligned} \frac{d\mathcal{H}(x_1(t),x_2(t))}{dt} =& \frac{\partial \mathcal{H}}{\partial x_1}\dot{x}_1 + \frac{\partial \mathcal{H}}{\partial x_2}\dot{x}_2 \\ =& \frac{\partial \mathcal{H}}{\partial x_1}\frac{\partial \mathcal{H}}{\partial x_2} - \frac{\partial \mathcal{H}}{\partial x_2}\frac{\partial \mathcal{H}}{\partial x_1} \\ =& 0. \end{aligned} \tag{66}$$

Note also that the vector field $\boldsymbol{f}(x_1, x_2)$ in (65) is divergence-free, i.e.,

$$\nabla \cdot \boldsymbol{f}(x_1,x_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial^2 \mathcal{H}}{\partial x_1 \partial x_2} - \frac{\partial^2 \mathcal{H}}{\partial x_1 \partial x_2}. \tag{67}$$

Therefore the flow gerated by (64) preserves volume in phase space. We emphasize that two-dimensional divergence-free vector fields are necessarily Hamiltonian and therefore conservative.

However, conservative systems are not necessarily Hamiltonian, nor volume preserving. It is rather straightforward indeed to manufacture systems that conserve specific quantities but are not Hamiltonian. As an example, consider the function

$$E(x_1,x_2) = x_1^2 x_2 + x_2. \tag{68}$$

A two-dimensional dynamical systems conserves $E(x_1, x_2)$ along a trajectory if an only if

$$\frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2 = 0. \tag{69}$$

Substituting equation (68) into (69) yields

$$2x_1x_2\dot{x}_1 + (x_1^2 + 1)\dot{x}_2 = 0. \tag{70}$$

At this point it is clear that the system

$$\begin{cases} \dot{x}_1 = -x_2^2 \\ \dot{x}_2 = \dfrac{2x_1x_2^3}{x_1^2 + 1} \end{cases} \tag{71}$$

conserves (68) along any trajectory (and therefore it is a conservative system), but it is not divergence-free nor Hamiltonian. Moreover, it is clear that there exits an infinite number of dynamical systems conserving (68). Another one is

$$\begin{cases} \dot{x}_1 = -\sin(x_2) \\ \dot{x}_2 = \dfrac{2x_1x_2\sin(x_2)}{x_1^2 + 1} \end{cases} \tag{72}$$

which clearly satisfies condition (70).

Appendix A: Derivation of the Hamilton's equations

The differential of the Hamilton's function $\mathcal{H}(\boldsymbol{p}, \boldsymbol{q}, t)$ can be written as

$$d\mathcal{H}(\boldsymbol{p}, \boldsymbol{q}, t) = \sum_{i=1}^{n} \left(\frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial q_i} dq_i \right) + \frac{\partial \mathcal{H}}{\partial t} dt. \tag{73}$$

On the other hand, by using the definition (44) we obtain

$$d\mathcal{H}(\boldsymbol{p}, \boldsymbol{q}, t) = \sum_{i=1}^{n} \left(dp_i \dot{q}_i + p_i d\dot{q}_i - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt. \tag{74}$$

Using the Euler-Lagrange equations (16) and the definition of canonical momenta (40) we have

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \dot{p}_i. \tag{75}$$

Setting the equality between (73) and (74), and taking into account (75) yields

$$\sum_{i=1}^{n} \left[\left(\frac{\partial \mathcal{H}}{\partial p_i} - \dot{q}_i \right) dp_i + \left(\frac{\partial \mathcal{H}}{\partial q_i} + \dot{p}_i \right) dq_1 \right] + \left(\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{L}}{\partial t} \right) dt = 0. \tag{76}$$

Assuming that dp_i, dq_i and dt are arbitrary, we obtain

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad \text{and} \qquad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{77}$$

Which coincide with the Hamilton's equations of motion (45).

Next, we prove that if the Lagrangian function does not depend explicitly on time then the Hamiltonian is conserved along trajectories. To this end, we first notice that if $\mathcal{L}$ does not explicitly depend on time then $\mathcal{H}$ does not explicitly dependent on time (by the last equation in (77)). Evaluating $\mathcal{H}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ along a trajectory of the system yields

$$\mathcal{H}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) = \sum_{i=1}^{n} \frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)}{\partial \dot{q}_i} \dot{q}_i(t) - \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)). \tag{78}$$

Differentiating with respect to time yields

$$\frac{d}{dt} \mathcal{H}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) = \sum_{i=1}^{n} \frac{d}{dt} \left(\frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)}{\partial \dot{q}_i} \dot{q}_i(t) \right) - \frac{d}{dt} \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)). \tag{79}$$

Note that

$$\begin{aligned}
\frac{d}{dt} \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) &= \sum_{i=1}^{n} \left(\frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))}{\partial q_i} \dot{q}_i(t) + \frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))}{\partial \dot{q}_i} \ddot{q}_i(t) \right) \\
&= \sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))}{\partial \dot{q}_i} \dot{q}_i(t) + \frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))}{\partial \dot{q}_i} \ddot{q}_i(t) \right) \\
&= \sum_{i=1}^{n} \frac{d}{dt} \left(\frac{\partial \mathcal{L}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))}{\partial \dot{q}_i} \dot{q}_i(t) \right),
\end{aligned} \tag{80}$$

where in the second equality we used the Euler-Lagrange equations (16). Substituting (80) into (79) yields

$$\frac{d}{dt} \mathcal{H}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) = 0, \tag{81}$$

which proves that the Hamiltonian is conserved along trajectories.

Nöether's theorem and conservation laws.

Any symmetry of the action functional (6), i.e., an invariance under a transformation group, is associated with a conservation law for the system. This remarkable result was proved by Emmy Nöether in 1918. To formalize Nöether's theorem, let us consider one-parameter group of transformations generated by the dynamical system

$$\begin{cases} \dfrac{d\boldsymbol{Q}(\epsilon,\boldsymbol{q})}{d\epsilon} = \boldsymbol{G}\left(\boldsymbol{Q}(\epsilon,\boldsymbol{q})\right) \\ \boldsymbol{Q}(0,\boldsymbol{q}) = \boldsymbol{q} \end{cases} \tag{82}$$

where ϵ parameterizes the transformation. The vector field $\boldsymbol{G}$ generates the flow $\boldsymbol{Q}(\epsilon,\boldsymbol{q})$. For small ϵ we have the infinitesimal transformation

$$\boldsymbol{Q}(\epsilon,\boldsymbol{q}) = \boldsymbol{q} + \epsilon\boldsymbol{G}(\boldsymbol{q}). \tag{83}$$

Setting up the condition for the action functional (6) to be invariant under the transformation group (83)-(91) yields

$$\mathcal{A}\left([\boldsymbol{Q}(\epsilon,\boldsymbol{q})]\right) = \mathcal{A}\left([\boldsymbol{q}]\right). \tag{84}$$

The first order condition that follows from (84) is

$$\frac{d}{d\epsilon}\,\mathcal{A}\left([\boldsymbol{q}+\epsilon\boldsymbol{Q}(\boldsymbol{q})]\right)\Big|_{\epsilon=0} = 0. \tag{85}$$

This can be written as

$$\int_{t_1}^{t_2} \frac{d}{d\epsilon}\,\mathcal{L}\left(\boldsymbol{q}+\epsilon\boldsymbol{G}(\boldsymbol{q}), \dot{\boldsymbol{q}}+\epsilon\frac{d\boldsymbol{G}(\boldsymbol{q}(t))}{dt}, t\right)\Bigg|_{\epsilon=0} dt = 0, \tag{86}$$

i.e.,

$$\begin{aligned} 0 &= \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{G}(\boldsymbol{q}) + \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\frac{d\boldsymbol{G}(\boldsymbol{q}(t))}{dt}\right)dt \\ &= \int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}}\cdot\boldsymbol{G}(\boldsymbol{q}) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q})\right)dt + \left[\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q}(t))\right]_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2}\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\right)}_{=\boldsymbol{0}\text{ by Eq. (16)}}\cdot\boldsymbol{G}(\boldsymbol{q})dt + \left[\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q}(t))\right]_{t_1}^{t_2} \\ &= \left[\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q}(t))\right]_{t_1}^{t_2}. \end{aligned} \tag{87}$$

This implies that the quantity

$$\Lambda = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q}) \tag{88}$$

is a constant of motion for the system[5], i.e., it is a conserved quantity. Such conserved quantities are also called *first integrals* of the system.

[5]Note that (87) implies that

$$\frac{\partial\mathcal{L}\left(\boldsymbol{q}(t_2),\dot{\boldsymbol{q}}(t_2),t_2\right)}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q}(t_2)) = \frac{\partial\mathcal{L}\left(\boldsymbol{q}(t_1),\dot{\boldsymbol{q}}(t_1),t_1\right)}{\partial\dot{\boldsymbol{q}}}\cdot\boldsymbol{G}(\boldsymbol{q}(t_1)) \tag{89}$$

for arbitrary t_1 and t_2.

More generally, it is possible to consider a transformation group that involves both the configuration variables $\boldsymbol{q}$, i.e., (83) as well as the time variable t via[6]

$$\tau(\epsilon, t) = t + \epsilon T(t). \tag{91}$$

The perturbed action functional in this case is

$$\mathcal{A}\left([\boldsymbol{q} + \boldsymbol{G}(\boldsymbol{q})]\right) = \int_{t_1+\epsilon T(t_1)}^{t_2+\epsilon T(t_2)} \mathcal{L}\left(\boldsymbol{q} + \epsilon \boldsymbol{G}(\boldsymbol{q}), \dot{\boldsymbol{q}} + \epsilon \frac{d\boldsymbol{G}(\boldsymbol{q}(t))}{dt}, t + \epsilon T(t)\right) dt. \tag{92}$$

By following the same steps that lead us to (87), i.e., by developing equation (85), it can be shown that invariace of (92) under the transformation (83)-(91) is equivalent to conservation of the following quantity

$$\Lambda = \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \dot{\boldsymbol{q}} - \mathcal{L}\right) T(t) + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \boldsymbol{G}(\boldsymbol{q}). \tag{93}$$

Energy conservation. If $\mathcal{L}$ does not depend explicitly on time, then of course $\mathcal{L}$ is invariant under the time translation $t \to t + \epsilon$. Such a time translation is easily generated by setting $T(t) = 1$ in (91) and (93) (together with $\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}$). Hence, if $\mathcal{L}$ does not depend explicitly on time then

$$\mathcal{H} = \sum_{k=1}^{n} \frac{\partial \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{q}_k} \dot{q}_k - \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{94}$$

is a constant of motion, i.e., it is conserved along trajectories of the system. The function $\mathcal{H}$ is called *Hamiltonian* of the system.

[6] The infinitesimal time-warping transformation (91) is defined by the dynamical system

$$\begin{cases} \dfrac{d\tau(\epsilon, t)}{d\epsilon} = T\left(\tau(\epsilon, t)\right) \\ \tau(0, t) = t \end{cases} \tag{90}$$