Two dimensional nonlinear systems: global theory

We have seen that a great deal of quantitative information about nonlinear dynamical systems can be obtained can be obtained by "zooming-in" on a small portion of the phase space. Examples of such local analysis are the Hartman-Grobman linearization theorem , the existence of closed trajectories for conservative systems nearby relative maxima and minima of the energy function, etc. We have also discussed results for nonlinear systems that involve a large portion of the phase space. These results go under the umbrella of "global theory" of dynamical systems (see L. Perko, *"Differential equations and dynamical systems"*, Chapter 3), and their proof are often based on topological arguments. Examples of results from global theory are: the center manifold theorem for fixed points[1], the theorem on the existence and uniqueness of global solutions for dynamical systems, etc.

It is usually harder to obtain information about the dynamics of nonlinear systems on large portions of the phase space. For instance, can you tell if a dynamical systems has a limit cycle just by looking at the differential equations? However, for two-dimensional systems of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \tag{1}$$

it is possible to devise theorems that provide global information about the dynamics, i.e., statements about the dynamics within large portions of the phase space. In this course note we discuss some results of such "global theory", i.e.,

a) Index theory,

b) The Poincaré-Bendixson theorem,

c) The Bendixson criterion to rule out limit cycles,

d) Liénard systems.

All these results hold for two-dimensional dynamical systems, i.e., there is no Poincaré-Bendixson theorem or index theory for higher dimensional systems.

Index theory. Index theory provides global information about phase portraits of two-dimensional dynamical systems. In particular, it allows us to answer questions such as:

- Is there a fixed point inside a closed orbit?
- What types of fixed points can coalescence in a bifurcation?
- Is a certain phase portrait plausible or impossible?

The "index" is a number that is attached to a two-dimensional closed non-intersecting curve (i.e. a Jordan curve) placed in the phase space. Such number represents the number of counterclockwise rotations of the vector field $\boldsymbol{f}(\boldsymbol{x}) = (f_1(x_1, x_2), f_2(x_1, x_2))$ as we proceed counterclockwise along the curve and complete one loop. Let us first define what a Jordan curve is.

Definition 1 (Jordan curve)**.** A Jordan curve is a closed curve in $\mathbb{R}^2$ that is homeomorphic to the unit circle. This means that there exists an invertible transformation (continuous with continuous inverse) that maps the unit circle into the Jordan curve and back (see Figure 1).

[1]There are versions of center manifold theorems for attracting sets other than fixed points, e.g., periodic orbits (see L. Perko, *"Differential equations and dynamical systems"*, Section §3.5.

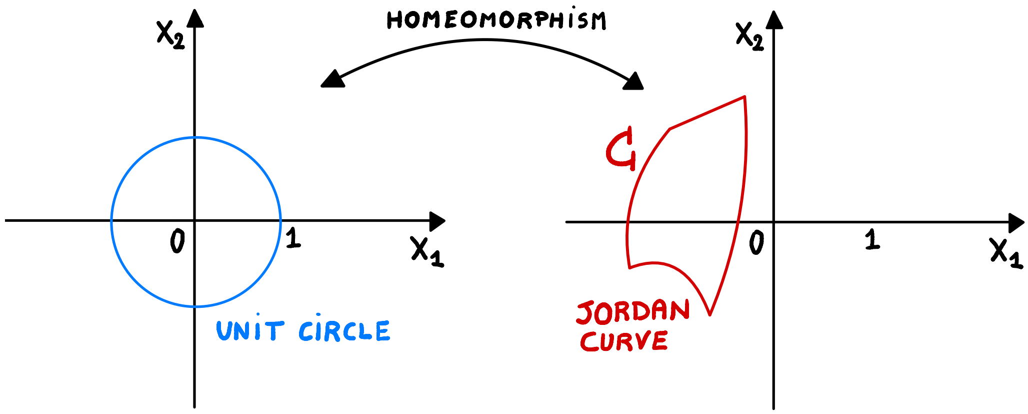


Figure 1: Sketch of a Jordan curve. A Jordan curve cannot have have intersections and it is not necessarily differentiable. For instance, it can be just piecewise continuous.

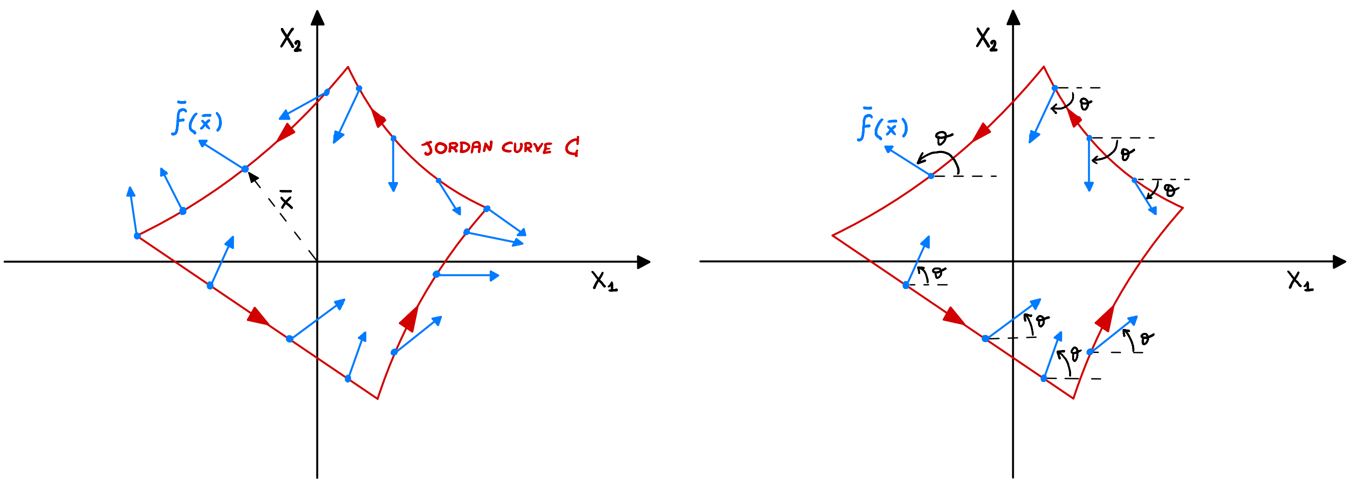


Figure 2: The index of a Jordan curve is the number of counterclockwise rotations of the vector field $\boldsymbol{f}(\boldsymbol{x}) = (f_1(x_1, x_2), f_2(x_1, x_2))$ in (1) as we loop along the curve counterclockwise and complete one loop.

Based on this definition it is clear that a Jordan curve cannot have intersections with itself, and moreover it is not necessarily differentiable. An example of a Jordan curve is shown in Figure 1.

Definition 2 (Index of a Jordan curve)**.** Let C be a Jordan curve. The index of C is the number of counterclockwise rotations of the vector field $(f_1(x_1, x_2), f_2(x_1, x_2))$ in (1) as we loop along the curve counterclockwise and complete one round trip.

In Figure 2 we sketch the graphical process of calculating the index of Jordan curve. To calculate the index of C, we simply integrate the angle ϑ (in radiants) of $\boldsymbol{f}(\boldsymbol{x})$ relative to the horizontal axis as we proceed along the curve C, i.e.,

$$I_C = \frac{1}{2\pi} \oint d\vartheta. \tag{2}$$

The infinitesimal angle $d\vartheta$ can be expressed as

$$d\vartheta = \frac{d}{d\lambda} \left[\arctan \left(\frac{f_2(s_1(\lambda), s_2(\lambda))}{f_1(s_1(\lambda), s_2(\lambda))} \right) \right] d\lambda, \tag{3}$$

where $\boldsymbol{s}(\lambda) = (s_1(\lambda), s_2(\lambda))$ is a parameterization of the Jordan curve. As an example, the unit circle is a

Jordan curve that can be parameterized as

$$s_1(\lambda) = \cos(2\pi\lambda), \qquad s_2(\lambda) = \sin(2\pi\lambda), \qquad \lambda \in [0,1]. \tag{4}$$

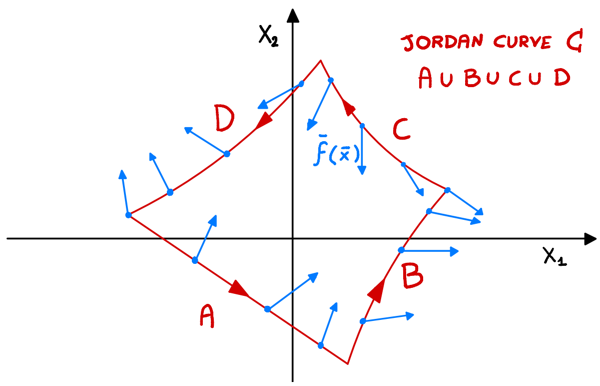
By developing the expression (3) we obtain

$$\begin{aligned}
d\vartheta =& \frac{d}{d\lambda}\left[\arctan\left(\frac{f_2(s_1(\lambda), s_2(\lambda))}{f_1(s_1(\lambda), s_2(\lambda))}\right)\right] d\lambda \\
=& \frac{1}{1+\left(\dfrac{f_2}{f_1}\right)^2}\left[\frac{\partial}{\partial x_1}\left(\frac{f_2}{f_1}\right)\frac{ds_1}{d\lambda} + \frac{\partial}{\partial x_2}\left(\frac{f_2}{f_1}\right)\frac{ds_2}{d\lambda}\right] \\
=& \frac{1}{f_1^2+f_2^2}\left[f_1\left(\frac{\partial f_2}{\partial x_1}\frac{ds_1}{d\lambda} + \frac{\partial f_2}{\partial x_2}\frac{ds_2}{d\lambda}\right) - f_2\left(\frac{\partial f_1}{\partial x_1}\frac{ds_1}{d\lambda} + \frac{\partial f_1}{\partial x_2}\frac{ds_2}{d\lambda}\right)\right] d\lambda.
\end{aligned} \tag{5}$$

Hence, (2) can be written explicitly as

$$\boxed{I_C = \frac{1}{2\pi}\int_0^1 \frac{1}{f_1^2+f_2^2}\left[f_1\left(\frac{\partial f_2}{\partial x_1}\frac{ds_1}{d\lambda} + \frac{\partial f_2}{\partial x_2}\frac{ds_2}{d\lambda}\right) - f_2\left(\frac{\partial f_1}{\partial x_1}\frac{ds_1}{d\lambda} + \frac{\partial f_1}{\partial x_2}\frac{ds_2}{d\lambda}\right)\right] d\lambda} \tag{6}$$

for any closed curve, i.e., a curve satisfying $\boldsymbol{s}(0) = \boldsymbol{s}(1)$. As mentioned above, a Jordan curve can be the union of multiple smooth curves, e.g., four curves A, B, C, and D. In this case,



$$I_C = \frac{1}{2\pi}\left(\int_A d\vartheta + \int_B d\vartheta + \int_C d\vartheta + \int_D d\vartheta\right)$$

Example: Consider the two-dimensional vector field $\boldsymbol{f}(\boldsymbol{x})$ with components

$$f_1(x_1, x_2) = -x_2, \qquad f_2(x_1, x_2) = x_1. \tag{7}$$

Let us compute the index of the curve

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : \quad x_1^2 + x_2^2 = 1\}. \tag{8}$$

A simple parameterization of C is (see also Eq. (4))

$$s_1(\lambda) = \cos(2\pi\lambda), \qquad s_2(\lambda) = \sin(2\pi\lambda), \qquad \lambda \in [0,1]. \tag{9}$$

For this problem, equation (6) takes the form

$$\begin{aligned}
I_C =& \frac{1}{2\pi}\int_0^1 \frac{1}{s_1^2(\lambda)+s_1^2(\lambda)}\left[-s_2(\lambda)\frac{ds_1(\lambda)}{d\lambda} + s_1(\lambda)\frac{ds_2(\lambda)}{d\lambda}\right] d\lambda \\
=& \frac{1}{2\pi}\int_0^1 \frac{2\pi}{\sin^2(2\pi\lambda)+\cos^2(2\pi\lambda)}\left[\sin^2(2\pi\lambda)+\cos^2(2\pi\lambda)\right] d\lambda \\
=& 1.
\end{aligned} \tag{10}$$

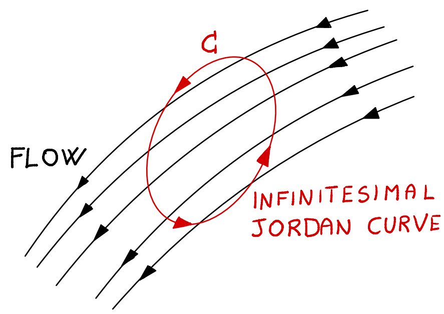


Figure 3: If there is no fixed point inside a Jordan curve C then the index of C is zero. In fact, we can continuously shrink C to an infinitesimal curve that sees the vector field $\boldsymbol{f}(\boldsymbol{x})$ parallel.

Recall that the vector field (7) defines a linear dynamical system with flow corresponding to a center. The trajectories of the system are circles rotating counterclockwise. This explains why the index of the curve C defined in (8) is equal to one.

There are several theorems characterizing the properties of index of a Jordan curve.

Theorem 1. The index of a Jordan curve C does not change if we continuously deform C without passing through any fixed points of the dynamical system (1).

The proof of the theorem relies on showing that an arbitrary infinitesimal deformation of the curve $\boldsymbol{s}(\lambda)$ leaves I_C unchanged, provided C does not intersect with a fixed point). Theorem 1 has several important consequences:

1. The index of a Jordan curve C does not depend on the curve, but rather it depends only on the fixed points enclosed by the curve.
2. If there is no fixed point inside the curve C then the index of C is zero. In fact, we can continuously shrink C to an infinitesimal circle that sees the vector field $\boldsymbol{f}(\boldsymbol{x})$ parallel (see Figure 3).
3. The index of any closed orbit of a two-dimensional system is one.

Since the index of a Jordan curve depends only on the fixed points enclosed by the curve, it makes sense to calculate the index of all jyperbolic fixed points we know of. To this end we pick an arbitrary Jordan curve enclosing such fixed pints and calculate the index. The following theorem summarizes the graphical findings shown in Figure 4.

Theorem 2 (Index of hyperbolic fixed points)**.** Let $\boldsymbol{x}^*$ be a hyperbolic fixed point of the two-dimensional nonlinear system (1). Then for any Jordan curve C enclosing $\boldsymbol{x}^*$

$$I_C = \text{sign}\left(\det(J_{\boldsymbol{f}}(\boldsymbol{x}^*)\right) = \text{sign}(\lambda_1\lambda_2), \tag{11}$$

where $J_{\boldsymbol{f}}(\boldsymbol{x}^*)$ is the Jacobian of the system (1) evaluated at $\boldsymbol{x}^*$, and λ_1, λ_2 are the eigenvalues of $J_{\boldsymbol{f}}(\boldsymbol{x}^*)$.

For *non-hyperbolic* fixed points the index I_C can be calculated using equation (2). It is easy to show that if a Jordan curve C encloses multiple fixed points then its index is the sum of the index.

Theorem 3. The index of a Jordan curve enclosing n fixed points is the sum of the indexes of the fixed points, i.e.,

$$I_C = I_1 + \cdots + I_n. \tag{12}$$

The proof if this theorem is based on the fact that we are free to shrink the black curve C to the curve shown hereafter (union of blue and red curves) without the changing the index. Clearly, the contributions

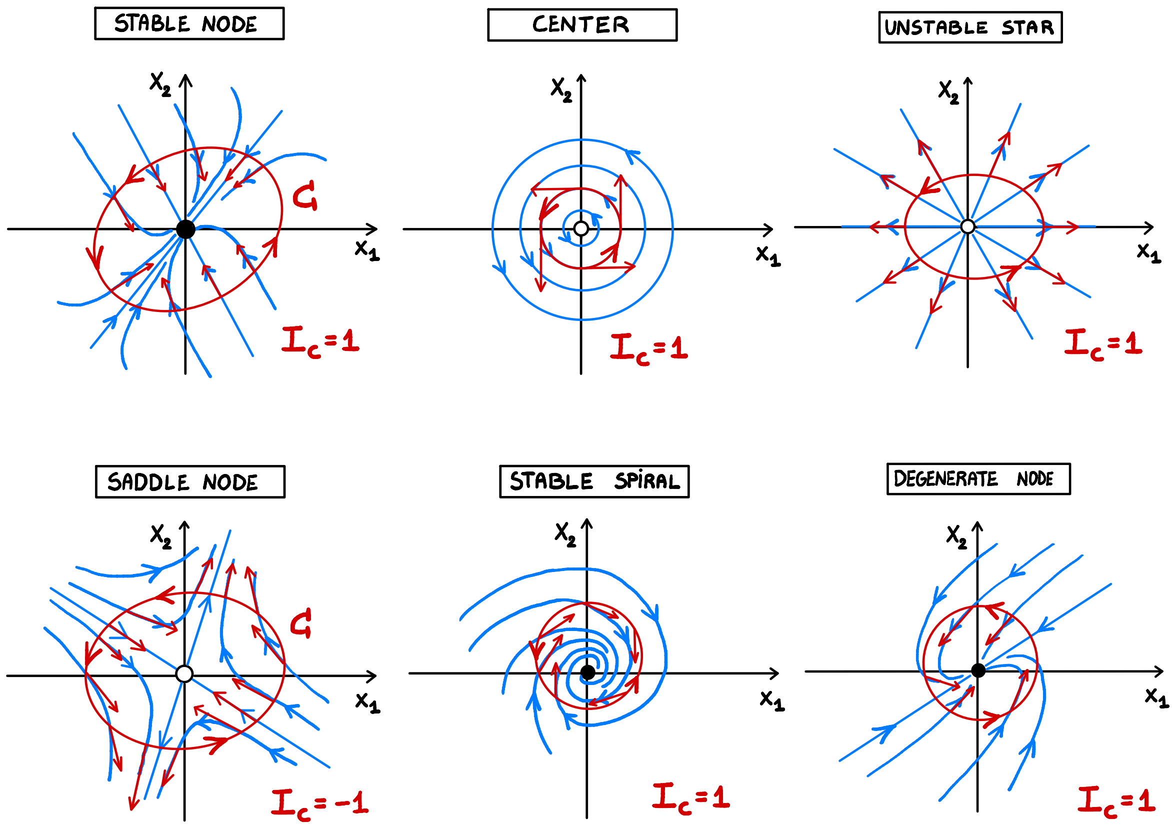
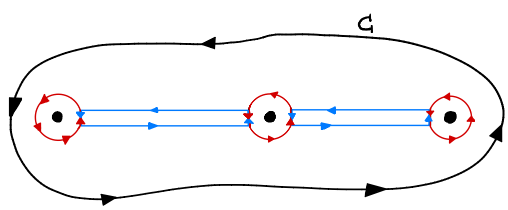


Figure 4: Index I_C of common fixed points. It is seen that $I_C = 1$ for spirals, stable and unstable nodes, centers, degenerated nodes and stars. $I_C = -1$ for saddle nodes.



to the index of the blue curves is zero as such curves do not include any fixed point. Hence, the index of C is essentially determined by the curves in red, which have index equal to the index of fixed point they enclose.

At this point one can draw all sorts of topological consequences related to the index. For example, a closed orbit of a two-dimensional nonlinear dynamical system must enclose fixed points with indexes that add up to one. In particular, we can have one spiral or one unstable node, but not just one saddle node[2]. It is possible however, that the closed orbit encloses one saddle node and two stable nodes. In fact the index of the sum in this case is $I_C = 2 - 1 = 1$, which is compatible with the closed orbit. In Figure 5 we sketch a plausible phase portrait of a closed orbit enclosing one saddle node and two stable nodes.

[2]Recall that the index of a saddle node is -1.

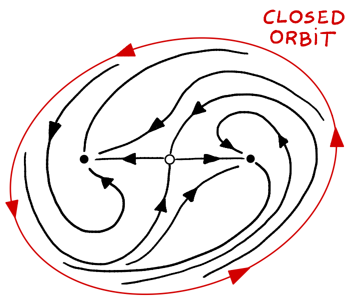


Figure 5: Closed orbit enclosing three fixed points (one saddle node and two stable nodes). The sum of the indexes of all fixed points enclosed by the closed orbit (one saddle node and two stable nodes) is one. It is impossible to have a closed orbit enclosing, e.g., only one saddle node and one stable node as the index of the curve in this case would be zero, contradicting the fact that closed orbits must have index one.

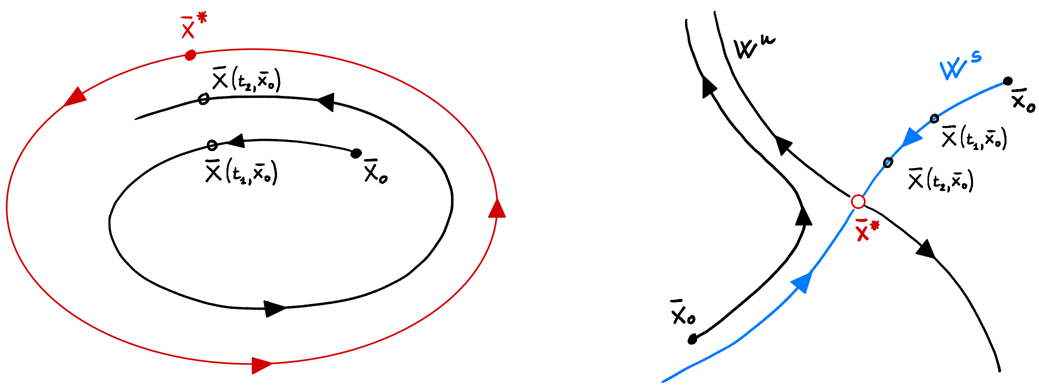


Figure 6: Shetch of two ω-limits. The need for a sequence of time instants $\{t_1, t_2, \dots\}$ in the definition of the ω-limit (Definition 3) is clearly explained with reference to the spiraling trajectory.

Poincaré-Bendixson theorem. The Poincaré-Bendixson theorem theorem gives us a complete determination of the asymptotic behavior of a large class of flows on the plane, cylinder, two-dimensional sphere, or more generally flows on smooth two-dimensional manifolds. The theorem, which was first conceived by Henri Poincaré in 1892 and finally proved by Ivar Bendixson in 1901, essentially rules out the existence of persistent (i.e., as $t \to \infty$) aperiodic orbits in bounded regions of any two-dimensional system.

To study the asymptotic behavior of the system (1) it is convenient to introduce the notion of ω-limit point of a trajectory and ω-limit sets.

Definition 3 (ω-limit set of a trajectory)**.** Let $\boldsymbol{X}(t, \boldsymbol{x}_0)$ be the flow generated by (1). A point $\boldsymbol{x}^*$ is called ω-limit of a trajectory with initial condition $\boldsymbol{x}_0$ if there exists a sequence of time instants[3] $\{t_1, t_2, \dots\}$ ($t_i \to \infty$) such that

$$\lim_{i \to \infty} \boldsymbol{X}(t_i, \boldsymbol{x}_0) = \boldsymbol{x}^*. \tag{13}$$

The set of all ω-limit points corresponding to a given $\boldsymbol{x}_0$ is called ω-limit set of the trajectory with initial condition $\boldsymbol{x}_0$ and it denoted as $\omega(\boldsymbol{x}_0)$

In Figure 6 we sketch a few ω limit sets corresponding to two-dimensional trajectories. More complex limit sets characterized by a finite number of fixed points connected by homoclinic or heteroclinic trajectories are are shown in Figure 7.

[3] Note that the sequence of time instants $\{t_1, t_2, \dots\}$ mentioned in the Definition 3 is not unique.

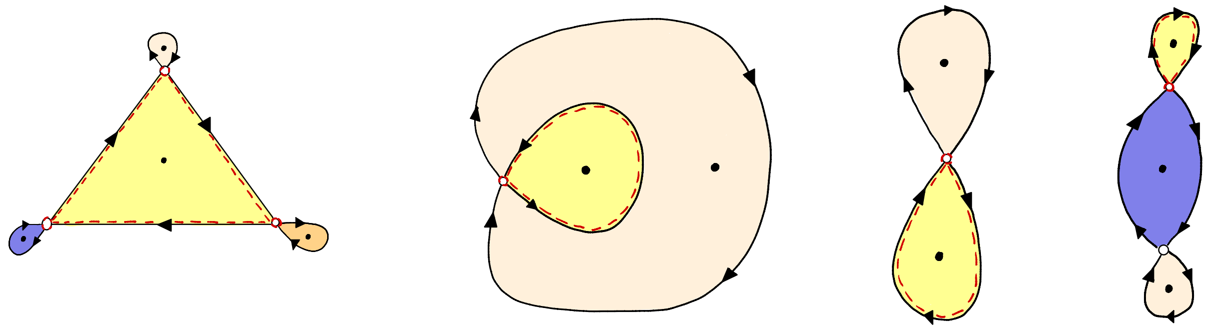


Figure 7: Sketch of ω-limit sets characterized by a finite number of fixed points connected by trajectories. The red dashed lines are ω-limit sets corresponding to initial conditions in the yellow domains, assuming that the fixed points represented by black dots are repellors.

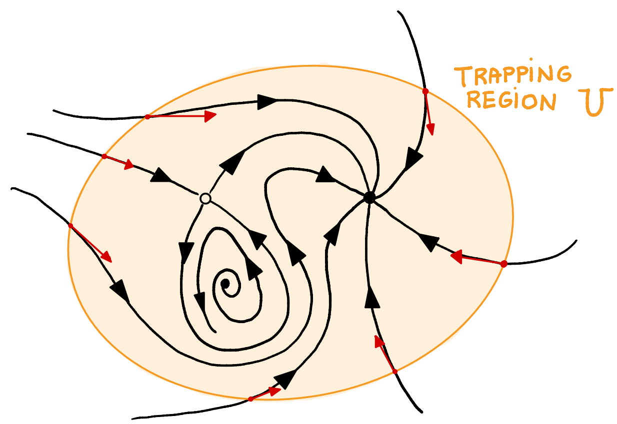


Figure 8: Sketch of a trapping region.

Definition 4 (Trapping region)**.** Let $\boldsymbol{X}(t, \boldsymbol{x}_0)$ be the flow generated by (1). A compact set $U \subseteq \mathbb{R}^2$ is called trapping region (or positively invariant set) if

$$\boldsymbol{X}(t, U) \subseteq U \qquad \text{for all } t \geq 0. \tag{14}$$

In other words, a trapping region $U \subseteq \mathbb{R}^2$ is a region from from which trajectories corresponding to $\boldsymbol{x}_0 \in U$ cannot escape U. This means that the vector field $\boldsymbol{f}(\boldsymbol{x})$ at the boundary of U is either tangent to the boundary of U, or it points inward.

If the boundary of the region U is given as a level set of some function $F(x_1, x_2)$, i.e., $F(x_1, x_2) = c$ then we can find a simple analytical condition for U to be a trapping region. In fact, we know that the gradient of F is orthogonal to the level sets of F. Moreover, if F is smaller inside U and larger outside U we have that ∇F points outward. In this case $U \subset \mathbb{R}^2$ is a trapping region if

$$\boldsymbol{f}(\boldsymbol{x}) \cdot \nabla F(\boldsymbol{x}) \leq 0, \tag{15}$$

for all points $\boldsymbol{x}$ on the level set of $F(\boldsymbol{x})$, i.e., for all points at the boundary of U.

Example: Let us show that the unit disk

$$U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \tag{16}$$

with boundary

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : \quad x_1^2 + x_2^2 = 1\} \tag{17}$$

is a trapping region for the system

$$\begin{cases} \dot{x}_1 = x_2^2 - x_1 \\ \\ \dot{x}_2 = -x_1 x_2 - x_2 \end{cases} \tag{18}$$

To this end, let us first write the circle C as the zero level set of the function

$$F(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

Note that $F(0,0) = -1$ and $F(2,0) = 3$. Therefore F increases as we go from the intererior of U towards its boundary C and beyond. The gradient $\nabla F = (2x_1, 2x_2)$ is orthogonal to C and in this case it points outward. Let us verify that the condition (15) is satisfied for the region (16) and the vector field (18). We have,

$$\boldsymbol{f}(\boldsymbol{x}) \cdot \nabla F(\boldsymbol{x}) = \dot{x}_1 \frac{\partial F}{\partial x_1} + \dot{x}_2 \frac{\partial F}{\partial x_2} = 2x_1(x_2^2 - x_1) + 2x_2(-x_1 x_2 - x_2) = -2(x_1^2 + x_2^2) < 0. \tag{19}$$

Hence U is a trapping region for the system (18). Note also that the system has a stable node at $(0,0)$.

The next question is, what kind of dynamics can we expect in the trapping region? For instance, can a trajectory in a trapping region wander around it forever without repeating itself (persistent aperiodic behavior)? This question is answered by the Poincaré-Bendixson theorem.

Theorem 4 (Poincaré-Bendixson)**.** Let $U \subseteq \mathbb{R}^2$ be a trapping region for the system (1) containing a finite number of fixed points $\{\boldsymbol{x}_1^*, \ldots, \boldsymbol{x}_n^*\}$. Let $\boldsymbol{x}_0 \in U$, and $\omega(\boldsymbol{x}_0)$ the ω-limit set of the trajectory with initial condition $\boldsymbol{x}_0$. Then one of the following possibilities holds:

1. $\omega(\boldsymbol{x}_0)$ is a fixed point;
2. $\omega(\boldsymbol{x}_0)$ is a closed orbit;
3. $\omega(\boldsymbol{x}_0)$ consists of a finite number of fixed points connected by orbits (see Figure 7).

Of course, if the trapping region has no fixed points in it then the only ω-limit set of a point $\boldsymbol{x}_0 \in U$ is an attracting closed orbit. Note that it is possible to have multiple closed orbits in a trapping region with no fixed points in it. In fact, Theorem 4 is not saying that the closed orbit is unique, but rather that a trajectory with initial condition $\boldsymbol{x}_0$ converges to a closed orbit if there are no fixed points in the trapping region. The closed orbit depends on $\boldsymbol{x}_0 \in U$ though. The next question is how many limit cycles[4] can you have in a bounded region of $\mathbb{R}^2$? This is answered by the following theorem due to the French mathematician Henri Dulac (1923).

Theorem 5 (Dulac's theorem)**.** In any bounded region of the plane, a planar system (1) with $\boldsymbol{f}(\boldsymbol{x})$ analytic in $\mathbb{R}^2$ has at most a finite number of limit cycles. Any polynomial system has at most a finite number of limit cycles in $\mathbb{R}^2$.

Bendixson's criterion to rule out closed orbits. By using Gauss's divergence theorem in two-dimensions it is possible to develop a simple criterion to rule out periodic orbits in systems of the form (1). This end, consider a simply connected domain $D \subset \mathbb{R}^2$. Let γ be a closed orbit in D. Such closed orbit identifies a subdomain of D which is denoted by R in Figure 9. Since the vector field $\boldsymbol{f}(\boldsymbol{x})$ is tangent to γ

[4] A limit cycle is an isolated closed trajectory in phase space. A center is not a limit cycle.

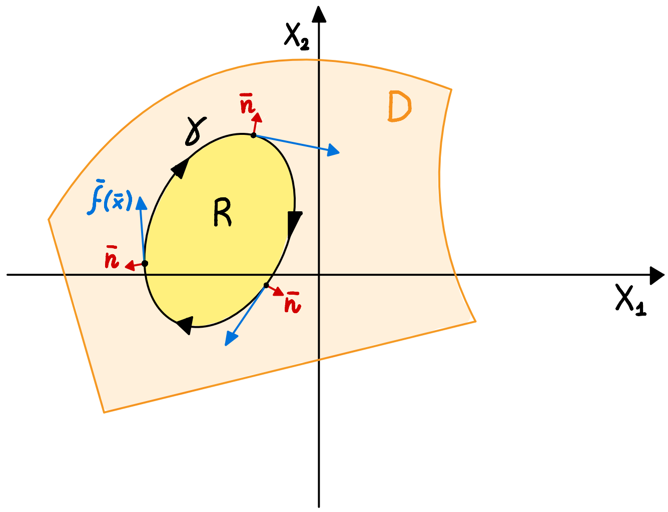


Figure 9: Illustration of the Bendixson's criterion to rule out periodic orbits in two-dimensional system. If the divergence of $\boldsymbol{f}(\boldsymbol{x})$ is not identically zero and does not change sign in the domain D then (1) has no closed orbits lying entirely in D. In this sketch the closed orbit is denoted by γ. The vector field $\boldsymbol{f}(\boldsymbol{x})$ is tangent to each point of γ.

at each point we have that the flux $\boldsymbol{f}(\boldsymbol{x})$ through γ (boundary of R) is zero. By using Gauss's divergence theorem we obtain

$$\int_R \nabla \cdot \boldsymbol{f}(\boldsymbol{x}) d\boldsymbol{x} = \underbrace{\int_0^1 \boldsymbol{f}(\boldsymbol{s}(\lambda)) \cdot \boldsymbol{n}(\boldsymbol{s}(\lambda)) d\lambda}_{=0} = 0, \tag{20}$$

where $\boldsymbol{s}(\lambda)$ is a parameterization of the closed curve γ, and $\boldsymbol{n}$ is the outward unit vector orthogonal to γ. Equation (20) implies the following theorem.

Theorem 6 (Bendixson's criterion)**.** If on a simply connected region $D \subseteq \mathbb{R}^2$ the divergence of the vector field $\boldsymbol{f}(\boldsymbol{x})$ is not identically zero and does not change sign, then (1) has no closed orbits lying entirely in D.

Example: The divergence of the system (18) is

$$\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -x_1 - 2. \tag{21}$$

Hence, by using Bendixson's criterion, we conclude that the system (18) cannot have periodic orbits lying entirely in any simply connected region D located in the half plane $x_1 \leq -2$, or the half plane $x_1 \geq -2$.

Liénard systems. The Poincaré-Bendixson Theorem can be used to establish the existence of limit cycles for certain two-dimensional systems, provided we can identify a trapping region with no fixed points in it. It is a far more delicate question to determine the exact number of limit cycles of a certain system or class of systems depending on parameters. In this section we discuss a classical result on the uniqueness of a stable limit cycle for systems of the form

$$\ddot{x} + h(x)\dot{x} + g(x) = 0. \tag{22}$$

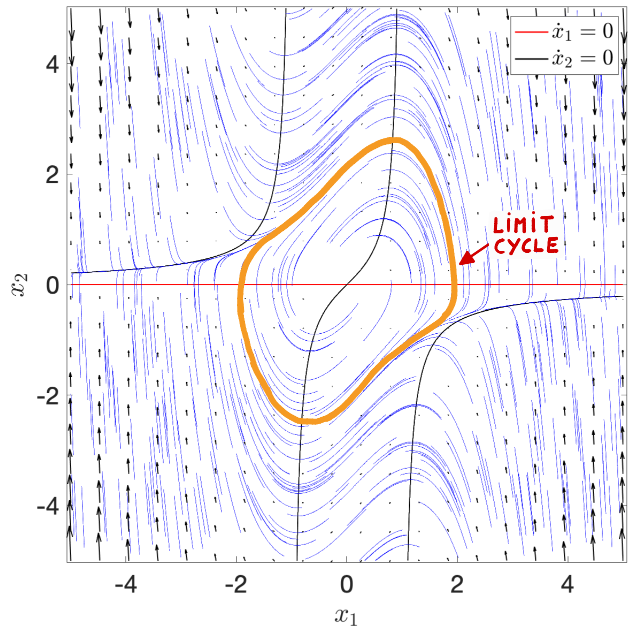


Figure 10: Phase portrait of the Van der Pol oscillator (24) for $\mu = 1$.

This result was first established by the French physicist Alfred-Marie Liénard in 1928. To formulate Liénard's theorem, let us first define

$$H(x) = \int_0^x h(z)dz. \tag{23}$$

Theorem 7 (Liénard's theorem)**.** The system (22) has exactly one stable limit cycle surrounding the origin of the system $(x, \dot{x}) = (0, 0)$ if the following conditions are satisfied:

1. $h, g \in C^1(\mathbb{R})$;
2. h is an even function $h(-x) = h(x)$, and g is an odd function $g(-x) = -g(x)$;
3. $g(x) > 0$ for all $x > 0$;
4. The function $H(x)$ in (23) has the following properties:
   - There unique zero $a > 0$ such that $H(a) = 0$
   - $H(x) < 0$ for all $0 < x < a$,
   - H increases monotonically to infinity for $x \geq a$.

Example (Van der Pol oscillator): The Van der Pol oscillator was originally proposed by the Dutch electrical engineer and physicist Van der Pol while he was working at Philips. The differential equation defining the oscillator is

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \qquad \mu > 0. \tag{24}$$

This is clearly a Liénard system. In fact the functions

$$h(x) = \mu(x^2 - 1) \qquad \text{and} \qquad g(x) = x, \tag{25}$$

satisfy all conditions listed in Theorem 7. Intuitively, the nonlinear term $\mu(x^2-1)\dot{x}$ is responsible for the creation of the limit cycle. In fact,

$$
\begin{aligned}
|x| < 1 \quad &\Rightarrow \quad \mu(x^2-1) < 0 \quad \Rightarrow \quad \text{negative damping} \quad \Rightarrow \quad \text{small oscillations amplified,}\\
|x| > 1 \quad &\Rightarrow \quad \mu(x^2-1) > 0 \quad \Rightarrow \quad \text{positive damping} \quad \Rightarrow \quad \text{large oscillations decay.}
\end{aligned}
$$

We can rewrite (24) as a two-dimensional system

$$
\begin{cases}
\dot{x}_1 = x_2\\
\dot{x}_2 = -\mu(x_1^2-1)x_2 - x_1
\end{cases}
\tag{26}
$$

In Figure 10 we plot the phase portrait generated by (26). As easily seen, the fixed point $(0,0)$ is an unstable spiral. Looking at the orbits of the system, we see that it is not straightforward to identify a trapping region in this case.