Bifurcations of equilibria in one-dimensional dynamical systems

We have already studied one-dimensional dynamical systems

$$\begin{cases} \dfrac{dx}{dt} = f(x) \\ x(0) = x_0 \end{cases} \tag{1}$$

and have thoroughly analyzed their properties. In particular, we established that such systems cannot exhibit trajectory reversals or oscillatory motion. Instead, the dynamics is fundamentally governed by the fixed points and their stability. In this course note, we study how the fixed points change when the function $f(x)$ in (1) depends on a real parameter $\mu \in \mathbb{R}$. To this end, we consider the dynamical system[1]

$$\begin{cases} \dfrac{dx}{dt} = f(x,\mu) \\ x(0) = x_0 \end{cases} \tag{2}$$

For each fixed value of μ, we can plot $f(x,\mu)$ versus x and identify any fixed points by locating the zeros of the function. Alternatively, we can interpret $f(x,\mu)$ as a real-valued function of two variables, i.e., as a surface embedded in three-dimensional Euclidean space.

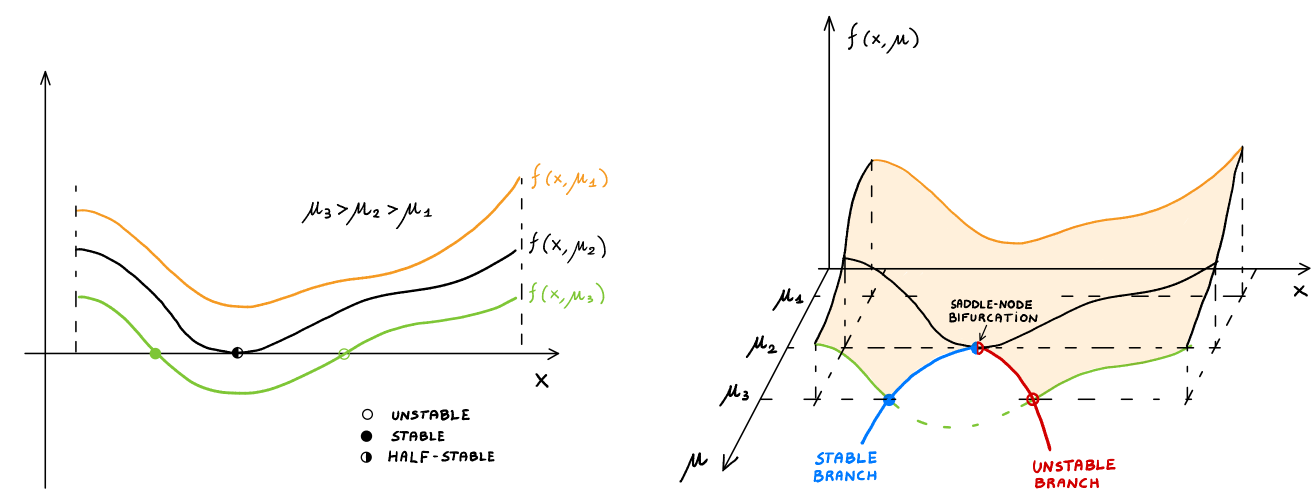

Figure 1: Sketch of a saddle-node bifurcation.

Clearly, the fixed points of the system (2) lie on the zero level set[2] of $f(x,\mu)$, i.e., they are solutions of the equation

$$f(x,\mu) = 0. \tag{4}$$

In Figure 1, the zero level set of $f(x,\mu)$ is represented by the stable and unstable branches of fixed points that emerge from a saddle-node bifurcation at $\mu = \mu_2$. Of course, there are many other ways in which the function $f(x,\mu)$ can intersect the (x,μ)-plane. For example, the zero level set may correspond to a subcritical pitchfork bifurcation. This bifurcation is illustrated in Figure 2.

[1] More generally, $f(x)$ can depend on multiple parameters, i.e., we may have $f(x,\mu_1,\ldots,\mu_M)$ in equation (2).

[2] The zero level set of a function $f(x,\mu)$ is the set of points $(x,\mu) \in \mathbb{R}^2$ for which the function vanishes, i.e.,

$$\{(x,\mu) \in \mathbb{R}^2 : \ f(x,\mu) = 0\} \qquad \text{(zero level set of } f). \tag{3}$$

If the function $f(x,\mu)$ does not intersect the (x,μ)-plane, then the zero level set is empty.

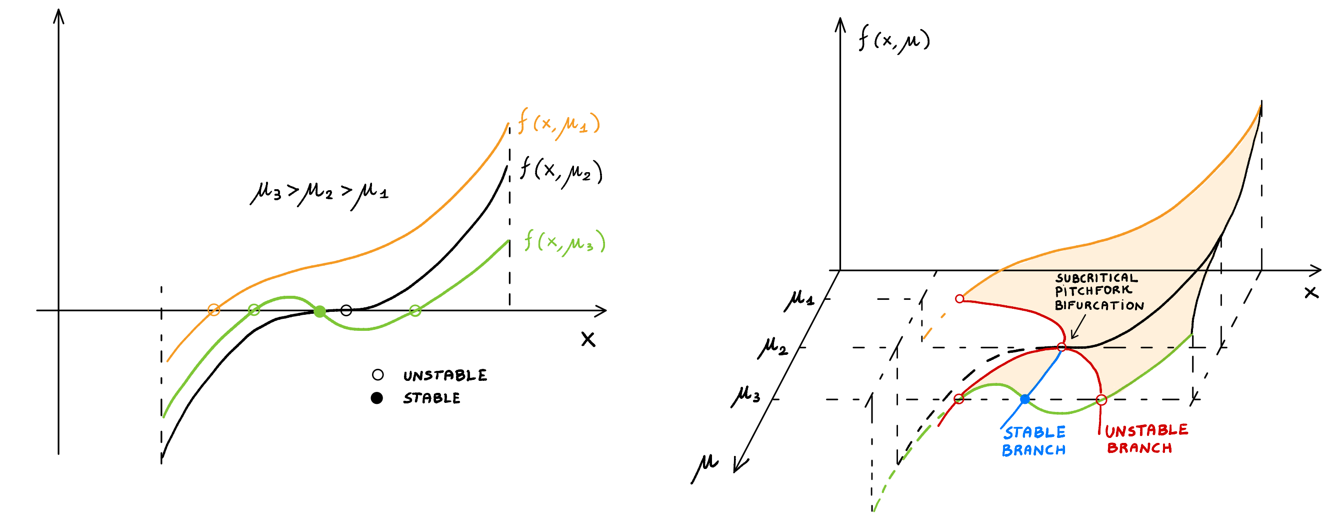

Figure 2: Sketch of a subcritical pitchfork bifurcation.

Bifurcation diagram. The zero level set of $f(x, \mu)$, i.e., the set of points $(x, \mu) \in \mathbb{R}^2$ satisfying equation (4), is referred to as the *bifurcation diagram* of fixed points. In practice, the bifurcation diagram provides the locations of all fixed points of the system as functions of the parameter μ. Hereafter, we sketch the bifurcation diagrams corresponding to the saddle-node bifurcation shown in Figure 1, and the subcritical pitchfork bifurcation illustrated in Figure 2.

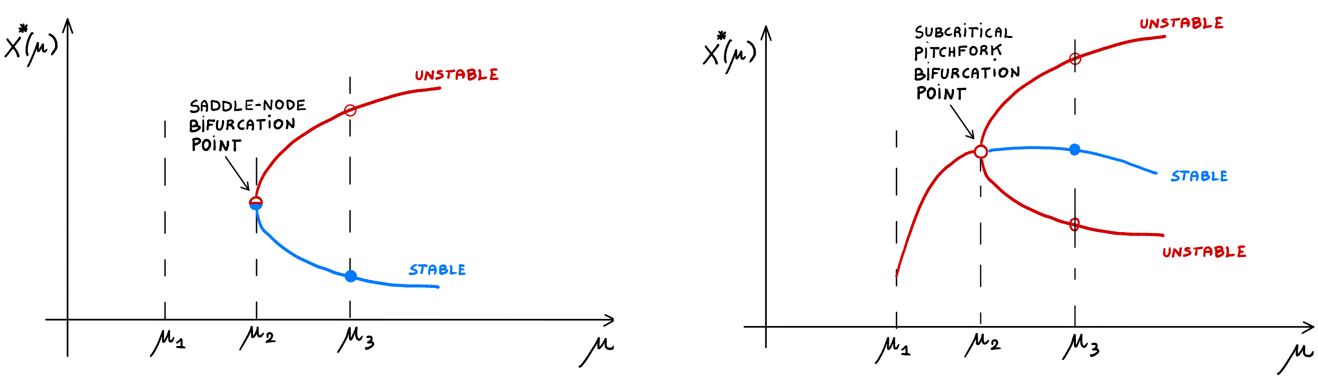

Figure 3: Bifurcation diagrams corresponding to the saddle-node bifurcation shown in Figure 1, and the subcritical pitchfork bifurcation sketched in Figure 2.

In a bifurcation diagram, we typically plot the locations of the fixed points $x^*(\mu)$ as functions of the bifurcation parameter μ. However, it is also possible to reverse the perspective and plot the bifurcation parameter $\mu(x^*)$ versus the fixed point locations x^*. In this formulation, the saddle-node bifurcation diagram shown in Figure 3(left) corresponds to a parabolic curve $\mu(x^*)$ with upward concavity.

What is the relationship between the coordinates of the fixed points x^* and the parameter μ? In particular, can the zero level set of $f(x, \mu)$ be expressed as the graph of a smooth function? This question is answered by the implicit function theorem.

Theorem 1 (Implicit function theorem)**.** Let $f(x,\mu)$ be a function of class $\mathcal{C}^1$ (i.e., continuously differentiable) in x and μ in a neighborhood of a point (x^*,μ^*) such that $f(x^*,\mu^*)=0$. If

$$\frac{\partial f(x^*,\mu^*)}{\partial x}\neq 0 \tag{5}$$

then there exists a neighborhood B of μ^* in which the zero level set of $f(x,\mu)$ can be represented as a graph of a smooth function $x^*(\mu)$, i.e.,

$$f(x^*(\mu),\mu)=0 \qquad \text{for all } \mu\in B. \tag{6}$$

The function $x^*(\mu)$ is of class $\mathcal{C}^1(B)$ (continuously differentiable in B) and it satisfies the additional properties:

$$x^*(\mu^*)=x^*, \qquad\qquad \frac{dx^*(\mu^*)}{d\mu}=-\frac{\dfrac{\partial f(x^*,\mu^*)}{\partial \mu}}{\dfrac{\partial f(x^*,\mu^*)}{\partial x}}. \tag{7}$$

Let us provide some comments:

- Theorem 1 implies that the bifurcation diagram consists of smooth curves $x^*(\mu)$, continuously differentiable with respect to μ, except at points (x^*,μ^*) where

$$\frac{\partial f(x^*,\mu^*)}{\partial x}=0. \tag{8}$$

- Property (7) follows by differentiating (6) with respect to μ and evaluating the result at $\mu=\mu^*$. Indeed,

$$f(x^*(\mu),\mu)=0 \quad\Rightarrow\quad \frac{d}{d\mu}f(x^*(\mu),\mu)=0 \quad\Rightarrow\quad \frac{\partial f}{\partial \mu}(x^*(\mu),\mu)+\frac{\partial f}{\partial x}(x^*(\mu),\mu)\frac{dx^*(\mu)}{d\mu}=0. \tag{9}$$

  Evaluating the last equation at $\mu=\mu^*$ and using $x^*(\mu^*)=x^*$, we obtain (7). Note that we can divide by $\partial f(x^*,\mu^*)/\partial x$ since it is nonzero by assumption (5).

- The roles of x and μ can be reversed in Theorem 1. That is, one can reformulate the implicit function theorem by treating x as the independent variable and μ as the dependent one. In this setting, if (x^*,μ^*) lies on the zero level set of f and $\partial f(x^*,\mu^*)/\partial\mu\neq 0$, then there exists a smooth function $\mu^*(x)$ defined in a neighborhood of x^* such that $f(x,\mu^*(x))=0$ for all x in that neighborhood. With reference to the saddle-node bifurcation shown in Figure 3, we see that at the bifurcation point we have $dx^*(\mu^*)/d\mu=\infty$, which suggests that

$$\frac{\partial f(x^*,\mu^*)}{\partial x}=0 \quad\text{and}\quad \frac{\partial f(x^*,\mu^*)}{\partial \mu}\neq 0 \quad \text{(see Eq. (7))}.$$

  On the other hand, $d\mu^*(x)/dx=0$, again implying that $\partial f(x^*,\mu^*)/\partial x=0$.

From this discussion, it appears that the conditions

$$f(x^*,\mu^*)=0, \qquad \frac{\partial f(x^*,\mu^*)}{\partial x}=0 \tag{10}$$

may indicate that a bifurcation occurs at the point (x^*,μ^*). While these conditions are indeed *necessary* for a bifurcation (since otherwise the implicit function theorem guarantees the local existence of a smooth

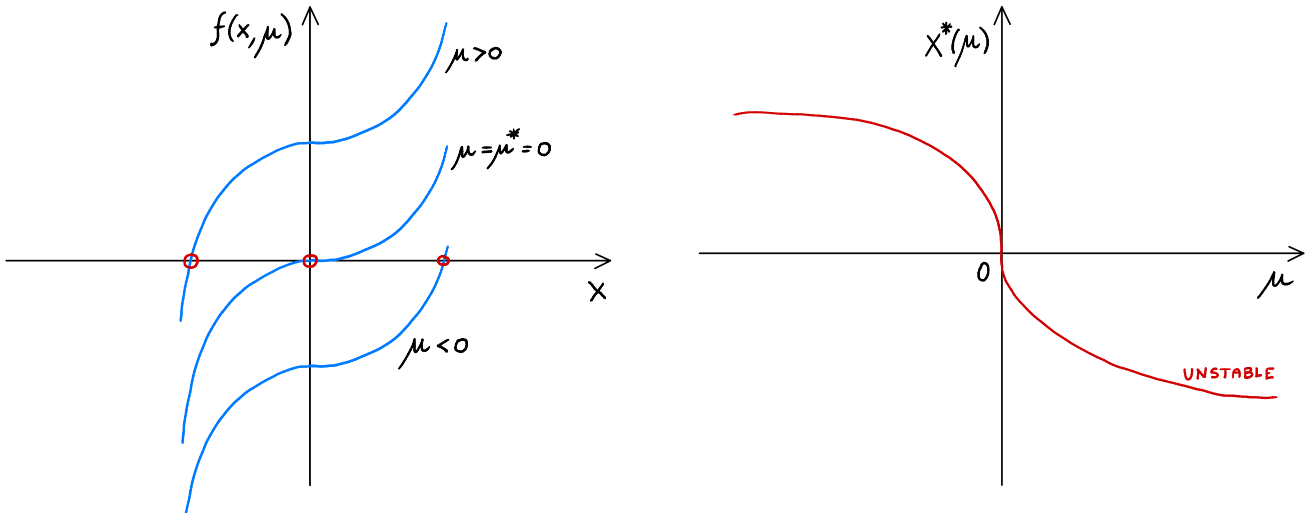

Figure 4: Sketch of the function (11) for different values of μ (left), and the corresponding bifurcation diagram (right). In this example, there is only one unstable fixed point, which shifts position as the parameter μ varies, but no actual bifurcation occurs. Nevertheless, the (necessary) conditions for a bifurcation given in equation (10) are both satisfied at the point $(x^*, \mu^*) = (0, 0)$.

branch of fixed points), they are not *sufficient* to ensure that a bifurcation actually takes place, as the following example clearly demonstrates.

Example: Consider the function

$$f(x, \mu) = x^3 + \mu. \tag{11}$$

Clearly

$$f(0, 0) = 0, \qquad \frac{\partial f(0, 0)}{\partial x} = 2x^2\big|_{(x,\mu)=(0,0)} = 0. \tag{12}$$

Hence, $(x^*, \mu^*) = (0, 0)$ may be a bifurcation point. However, the zero level set in this case can be expressed analytically as (see Figure 4)

$$f(x, \mu) = x^3 + \mu = 0 \qquad \Rightarrow \qquad x^*(\mu) = -\sqrt[3]{\mu}. \tag{13}$$

This shows that there is indeed no bifurcation at $(x^*, \mu^*) = (0, 0)$. In Figure 4 we sketch the function (11) for different values of μ, and the corresponding bifurcation diagram.

In general, a bifurcation is characterized by two or more branches of fixed points intersecting at a common location for some value of the bifurcation parameter μ (see Figure 5). The multiplicity of branches emerging from the bifurcation point (x^*, μ^*) is typically associated with the non-invertibility of the zero level set of $f(x, \mu)$ at (x^*, μ^*), a phenomenon that can be analyzed using a Taylor series expansion. Of course, the function $f(x, \mu)$ may exhibit much more intricate zero level sets, leading to multiple bifurcations of different types. For instance, Figure 5 sketches a bifurcation diagram that features five distinct bifurcations, including:

- Saddle-node bifurcation,
- Transcritical bifurcation,
- Pitchfork bifurcation (both supercritical and subcritical).

The final, more "exotic" bifurcation resembles a saddle-node but involves four branches rather than two.

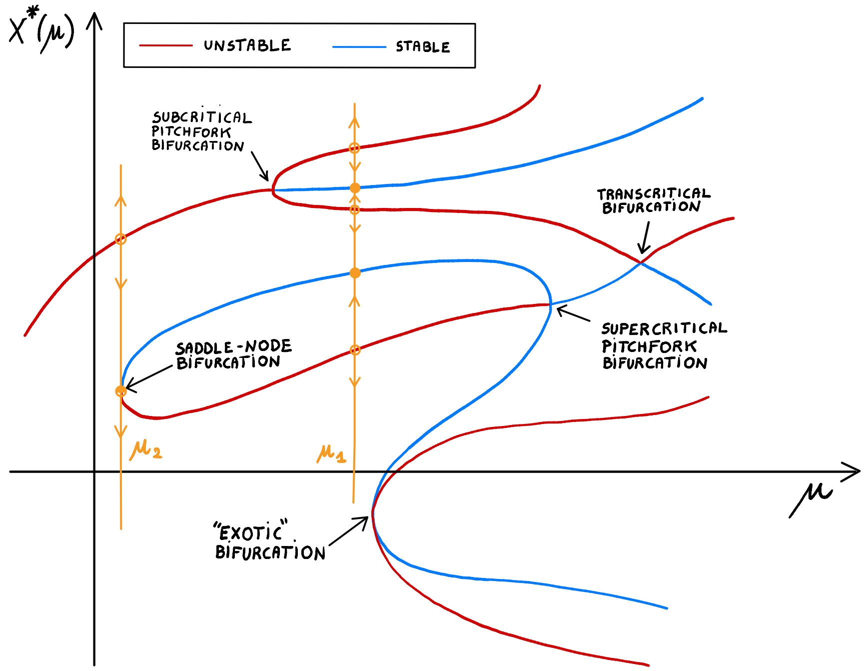

Figure 5: Sketch of a bifurcation diagram with five different bifurcations.

It is important to keep in mind that the bifurcation diagram represents the locations of the fixed points of the system as functions of the parameter μ. The continuity assumption on $f(x,\mu)$ imposes structural constraints on the diagram. In particular, fixed points of the same type – such as two stable nodes – cannot face each other. As a result, two stable branches or two unstable branches cannot appear facing one another in the bifurcation diagram.

Polynomial approximation of $f(x,\mu)$ at bifurcation points. To study the invertibility of the zero level set of $f(x,\mu)$ in a neighborhood of a bifurcation point (x^*,μ^*), we expand $f(x,\mu)$ in a Taylor series around (x^*,μ^*)

$$f(x,\mu) = \sum_{k,m=0}^{\infty} \frac{1}{k!\,m!} \frac{\partial^{k+m} f(x^*,\mu^*)}{\partial x^k \partial \mu^m}(x-x^*)^k(\mu-\mu^*)^m, \tag{14}$$

that is,

$$\begin{aligned}
f(x,\mu) = f(x^*,\mu^*) &+ \frac{\partial f(x^*,\mu^*)}{\partial x}(x-x^*) + \frac{\partial f(x^*,\mu^*)}{\partial \mu}(\mu-\mu^*) + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial x^2}(x-x^*)^2 \\
&+ \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial \mu^2}(\mu-\mu^*)^2 + \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu}(x-x^*)(\mu-\mu^*) \\
&+ \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3}(x-x^*)^3 + \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial \mu^3}(\mu-\mu^*)^3 \\
&+ \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x \partial \mu^2}(x-x^*)(\mu-\mu^*)^2 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^2 \partial \mu}(x-x^*)^2(\mu-\mu^*) + \cdots
\end{aligned} \tag{15}$$

We know that if (x^*,μ^*) is a bifurcation point then

$$f(x^*,\mu^*) = 0, \qquad \text{and} \qquad \frac{\partial f(x^*,\mu^*)}{\partial x} = 0. \tag{16}$$

A substitution of (16) into (15) yields

$$\begin{aligned} f(x,\mu) =& \frac{\partial f(x^*,\mu^*)}{\partial \mu} R + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial x^2} X^2 + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial \mu^2} R^2 + \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu} XR + \\ & \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3} X^3 + \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial \mu^3} R^3 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x \partial \mu^2} XR^2 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^2 \partial \mu} X^2 R + \cdots \end{aligned} \tag{17}$$

where we have introduced the "centered" variables

$$X = x - x^*, \qquad R = \mu - \mu^*. \tag{18}$$

If (x, μ) also lies on the zero level set, then $f(x, \mu) = 0$ in (17). This yields a polynomial equation in the variables X and R that characterizes the location of fixed points in a neighborhood of the bifurcation point (x^*, μ^*), which satisfies (16). The multiplicity of possible solutions to the polynomial equation

$$\begin{aligned} 0 =& \frac{\partial f(x^*,\mu^*)}{\partial \mu} R + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial x^2} X^2 + \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial \mu^2} R^2 + \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu} XR + \\ & \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3} X^3 + \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial \mu^3} R^3 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x \partial \mu^2} XR^2 + \frac{1}{2}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^2 \partial \mu} X^2 R + \cdots \end{aligned} \tag{19}$$

is precisely what gives rise to the presence of multiple branches of equilibria emanating from the bifurcation point (x^*, μ^*).

Depending on the leading-order terms in the polynomial expansion of the system near the bifurcation point, the number and arrangement of equilibrium branches involved in the bifurcation process can vary. As we shall see below, both saddle-node and transcritical bifurcations are locally represented by polynomials of degree 2, while pitchfork bifurcations correspond to polynomials of degree 3. The "exotic" bifurcation shown in Figure 5, involving four branches, is generated by a system that is locally equivalent to a polynomial of degree 4.

Saddle-node bifurcation. We now have all the necessary elements to characterize the saddle-node bifurcation illustrated in Figure 1 and in the left panel of Figure 3.

Theorem 2 (Saddle-node bifurcation)**.** Let (x^*, μ^*) be a fixed point of the dynamical system (2), i.e., $f(x^*, \mu^*) = 0$. If the following conditions are satisfied

$$\frac{\partial f(x^*,\mu^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*,\mu^*)}{\partial \mu} \neq 0, \qquad \frac{\partial^2 f(x^*,\mu^*)}{\partial x^2} \neq 0, \tag{20}$$

then the system undergoes a saddle-node bifurcation at (x^*, μ^*).

To characterize the saddle-node bifurcation we assume that R in equation (17) is of the same order of magnitude as X^2. For example, if $X \simeq 10^{-5}$, then $R \simeq 10^{-10}$. With this scaling and the conditions (20) we have that the leading terms in the Taylor expansion (17) have comparable magnitudes, allowing us to neglect higher-order terms in a systematic way. This allows us to write

$$f(x,\mu) \simeq AX^2 + BR + \cdots, \tag{21}$$

where A and B are shorthand for the corresponding second-order partial derivatives evaluated at (x^*, μ^*) i.e.,

$$B = \frac{\partial f(x^*,\mu^*)}{\partial \mu} \neq 0, \qquad \text{and} \qquad A = \frac{1}{2}\frac{\partial^2 f(x^*,\mu^*)}{\partial x^2} \neq 0. \tag{22}$$

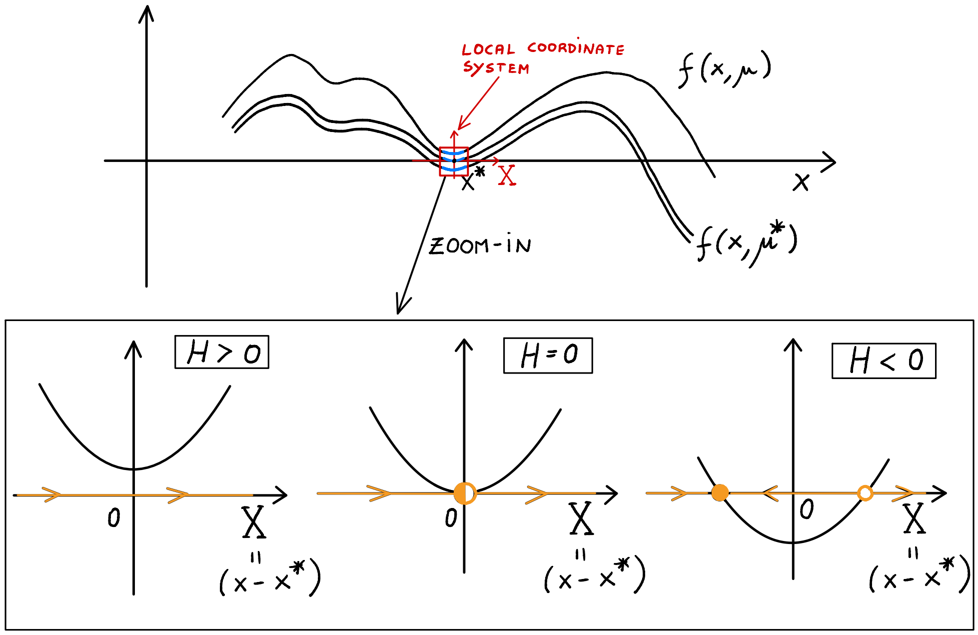

Figure 6: Saddle-node bifurcation. Shown is the function $f(x, \mu)$ for three values of μ, and a zoom-in of the bifurcation process. The nonlinear dynamical system (2) in such a small region is approximated by the normal form (26) (after appropriate rescaling).

Equation (21) allows us to write the following polynomial approximation[3] of the dynamical system (2) in a neighborhood of the bifurcation point (x^*, μ^*)

$$\frac{dX}{dt} = AX^2 + BR. \tag{24}$$

Diving by A and rescaling the time variable t as $\tau = At$ yields[4]

$$\frac{dX}{d\tau} = X^2 + H \qquad \text{(normal form)} \tag{26}$$

where $\tau = At$, and $H = RB/A$ is a rescaled version of the bifurcation parameter μ. Any dynamical system that undergoes a saddle-node bifurcation can be written as (26) in a neighborhood of the bifurcation point, i.e., for (x, μ) very close to (x^*, μ^*). This is the reason why (26) is called the *normal form* of a dynamical system that undergoes a saddle-node bifurcation. In Figure 6, we illustrate the meaning of the normal form of a saddle-node bifurcation. Figure 7 shows the corresponding bifurcation diagram, highlighting how the fixed points evolve as the parameter μ varies.

Example: Consider the nonlinear system

$$\frac{dx}{dt} = \sin(x) + \mu \tag{27}$$

[3]Note that since x^* is a constant we have

$$\frac{dX}{dt} = \frac{d(x - x^*)}{dt} = \frac{dx}{dt}. \tag{23}$$

[4]In equation (26) we assumed that $A > 0$. If $A <$ then we divide by the modulus of A, i.e., $|A|$, which leaves a minus sign in front of X^2 in (26), i.e.,

$$\frac{dX}{d\tau} = -X^2 + H \qquad , \qquad \tau = |A|t, \qquad , H = RB/|A|. \tag{25}$$

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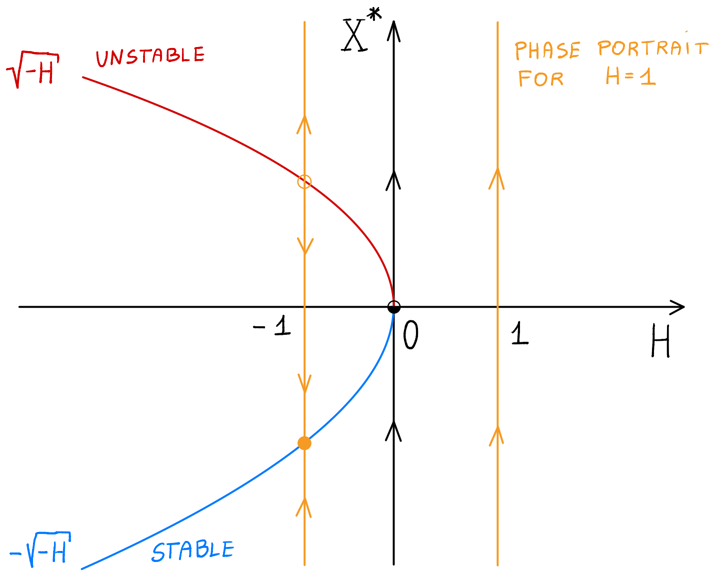

Figure 7: Bifurcation diagram for the normal form of a saddle-node bifurcation. Similarly to Figure 6, this bifurcation diagram describes what happens in an extremely small region that includes the bifurcation point (x^*, μ^*), i.e., the region in red in Figure 6.

In Figure 8 we plot $f(x, \mu)$ together with its zero level set, i.e., the bifurcation diagram. Note that if $\mu^* = 1$ we have that $f(x, \mu) = \sin(x) + \mu$ is tangent to the x axis at the points

$$x_k^* = -\frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}. \tag{28}$$

At such points we have

$$\frac{\partial f(x_k^*, \mu^*)}{\partial x} = \cos(x_k^*) = 0 \qquad \frac{\partial f(x_k^*, \mu^*)}{\partial \mu} = 1 \neq 0 \qquad \frac{\partial^2 f(x_k^*, \mu^*)}{\partial x^2} = -\sin(x_k^*) = -1 \neq 0. \tag{29}$$

Hence, the conditions of Theorem 2 are satisfied, implying that (x_k^*, μ^*) $(k \in \mathbb{Z})$ are all saddle-node bifurcations. It is straightforward to show that when $\mu^* = -1$ there is another infinite number of of saddle-node bifurcations at

$$x_k^* = \frac{\pi}{2} + 2k\pi, \qquad k \in \mathbb{Z}. \tag{30}$$

Example: Consider the nonlinear system

$$\frac{dx}{dt} = e^{-x^2/\mu} - \frac{\sin(x\mu)}{(x^2+1)}. \tag{31}$$

In this case, it is not possible to determine the fixed points of the system analytically. In fact, the fixed points are solutions to the transcendental equation

$$e^{-x^2/\mu} = \frac{\sin(x\mu)}{(x^2+1)}, \tag{32}$$

which cannot be solved analytically. However, it is rather straightforward to compute the fixed points numerically, e.g., as zero level sets of the two dimensional function (31) or using any root finding solver. The result is shown in Figure 9, where we see that there is an infinite number of saddle node bifurcations that tend to cluster as μ increases

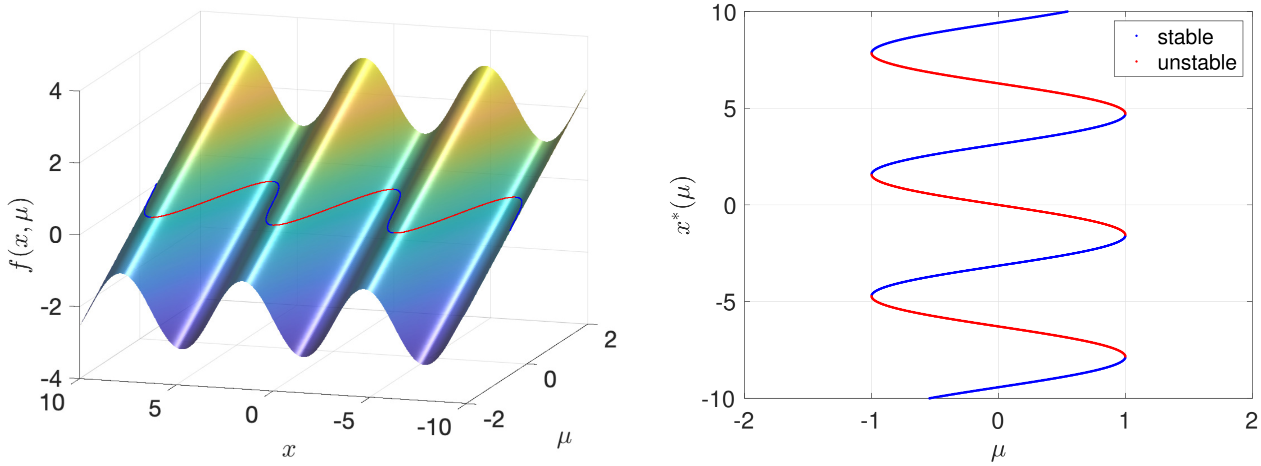

Figure 8: Plot of the right hand side of equation 27, i.e., $f(x,\mu) = \sin(x) + \mu$ together with its zero level set (left), and bifurcation diagram (right).

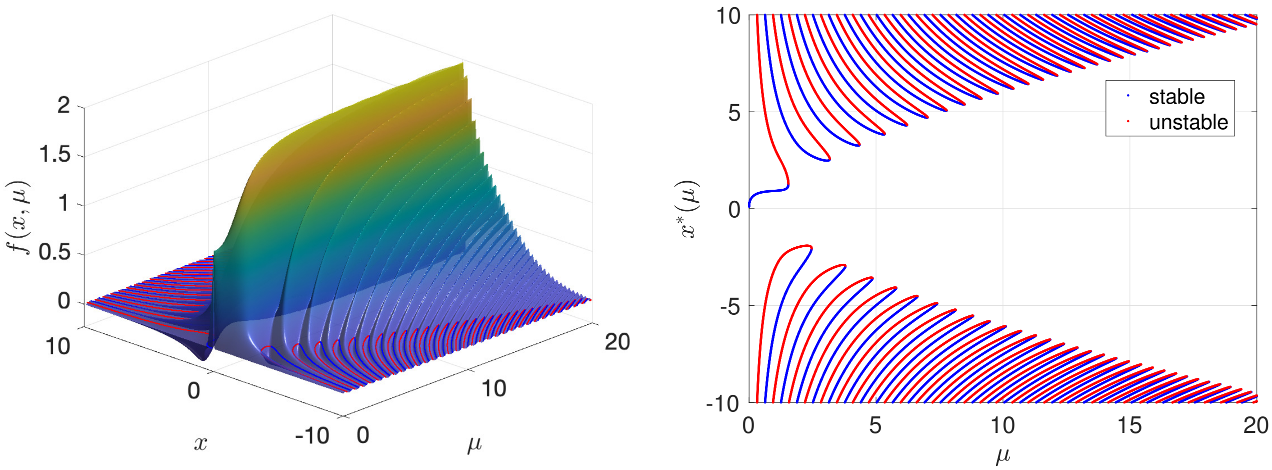

Figure 9: Plot of $f(x,\mu)$ defined in equation 31 (right hand side) together with its zero level set (left), and bifurcation diagram (right).

Transcritical bifurcation. Transcritical bifurcations are characterized by the following theorem.

Theorem 3 (Transcritical bifurcation)**.** Let (x^*, μ^*) be a fixed point of the dynamical system (2), i.e., $f(x^*, \mu^*) = 0$. If the following conditions are satisfied

$$\frac{\partial f(x^*, \mu^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*, \mu^*)}{\partial \mu} = 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} \neq 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} \neq 0 \tag{33}$$

then the system undergoes a transcritical bifurcation at (x^*, μ^*).

A substitution of (33) into (17) yields (to leading order in $X = x - x^*$ and $R = \mu - \mu^*$)

$$f(x,\mu) = BX^2 + CXR + \cdots . \tag{34}$$

In equation (34) we set

$$B = \frac{1}{2}\frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} \neq 0, \qquad \text{and} \qquad C = \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} \neq 0. \tag{35}$$

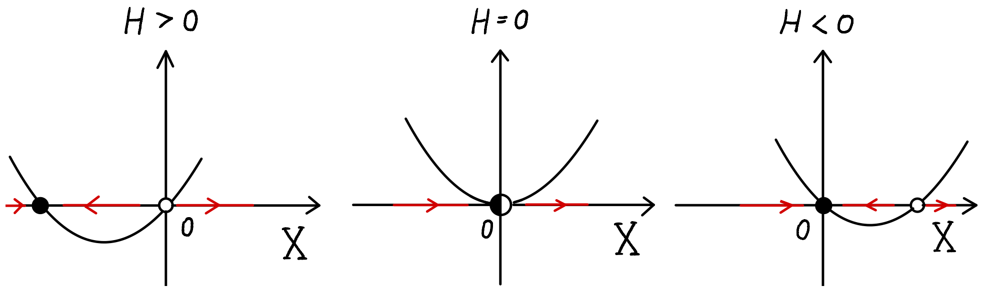

Figure 10: Transcritical bifurcation in local coordinates. Shown is the function $X^2 + XH$ appearing at the right hand side of the normal form (38) for three values of H. The nonlinear dynamical system (2) is approximated by the normal form (38) in a neighborhood of the bifurcation point (after appropriate rescaling).

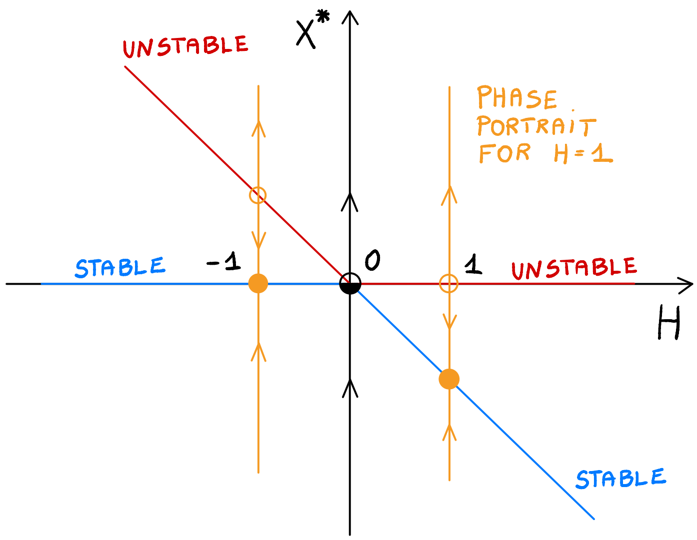

Figure 11: Bifurcation diagram for the normal form of a transcritical bifurcation.

Hence, the dynamics nearby a transcritical bifurcation point is characterized by the following polynomial approximation of the dynamical system (2)

$$\frac{dX}{dt} = BX^2 + CXR, . \tag{36}$$

which can be normalized (divide by B) as[5]

$$\frac{dX}{d\tau} = X^2 + XH \qquad \text{(normal form)}, \tag{38}$$

where $\tau = Bt$, and $H = CR/B$. The fixed points of (38) are $X^* = 0$ and $X^* = -H$. In Figure 10 plot the velocity vector that defines the normal form of a transcritical bifurcation and sketch the bifurcation diagram.

[5]As in the case of the saddle-node bifurcation, if $B < 0$ then we divide by the modulus of B, which yields a minus in front of X^2 in (36), i.e.,

$$\frac{dX}{d\tau} = -X^2 + XH. \tag{37}$$

Example: Consider the following dynamical system

$$\frac{dx}{dt} = \underbrace{\mu \log(x) + x - 1}_{f(x,\mu)}. \tag{39}$$

The fixed points are obtained by setting $f(x, \mu) = 0$. This yields,

$$\mu \log(x) = 1 - x. \tag{40}$$

Clearly, for $x = 1$ the equation above reads $0 = 0$, which means that $x^* = 1$ is a fixed point for all values of μ. Next, we compute the derivative of $f(x, \mu)$ with respect to x

$$\frac{\partial f(x, \mu)}{\partial x} = \frac{\mu}{x} + 1. \tag{41}$$

A necessary condition for (x^*, μ^*) to be a bifurcation point is

$$\frac{\partial f(x^*, \mu^*)}{\partial x} = 0 \qquad \Rightarrow \qquad \mu^* = -x^*. \tag{42}$$

Recalling that $x^* = 1$ is always a fixed point, we find that $(x^*, \mu^*) = (1, -1)$ could be a bifurcation point. Let us verify that $(1, -1)$ is indeed a transcritical bifurcation point. To this end, we just need to verify the conditions in Theorem 3. We have,

$$\frac{\partial f(1, -1)}{\partial \mu} = 0, \qquad \frac{\partial^2 f(1, -1)}{\partial x^2} = 1 \neq 0, \qquad \frac{\partial^2 f(1, -1)}{\partial x \partial \mu} = 1 \neq 0. \tag{43}$$

Therefore $(x^*, \mu^*) = (1, -1)$ is a transcritical bifurcation point. Let us compute the normal form of the system (39) at the bifurcation point. Recalling (35)-(36) and using (43) we have

$$\frac{dX}{dt} = \frac{X^2}{2} + XR, \tag{44}$$

where $X = x - 1$ and $R = \mu + 1$. Divide (44) by 1/2 to obtain the normal form

$$\frac{dX}{d\tau} = X^2 + XH, \tag{45}$$

where

$$\tau = \frac{t}{2}, \qquad \text{and} \qquad H = 2(\mu + 1). \tag{46}$$

Pitchfork bifurcation. Another type common bifurcation of equilibria is the pitchfork bifurcation, which can be *supercritical* or *subcrititical* (see Figure 2 and Figure 5). The following Theorem characterizes pitchfork bifurcations.

Theorem 4 (Pitchfork bifurcation)**.** Let (x^*, μ^*) be a fixed point of the dynamical system (2), i.e., $f(x^*, \mu^*) = 0$. If the following conditions are satisfied

$$\frac{\partial f(x^*, \mu^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*, \mu^*)}{\partial \mu} = 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} = 0, \qquad \frac{\partial^3 f(x^*, \mu^*)}{\partial x^3} \neq 0, \qquad \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} \neq 0 \tag{47}$$

then the system undergoes a pitchfork bifurcation at (x^*, μ^*).

Pitchfork bifurcations can be supercritical or subcritical, depending on the sign of $\partial^3 f/\partial x^3$ and $\partial^2 f/\partial x \partial \mu$. A substitution of (47) into (17) yields (to leading order in $X = x - x^*$ and $R = \mu - \mu^*$)

$$f(x, \mu) = DX^3 + CXR + \cdots. \tag{48}$$

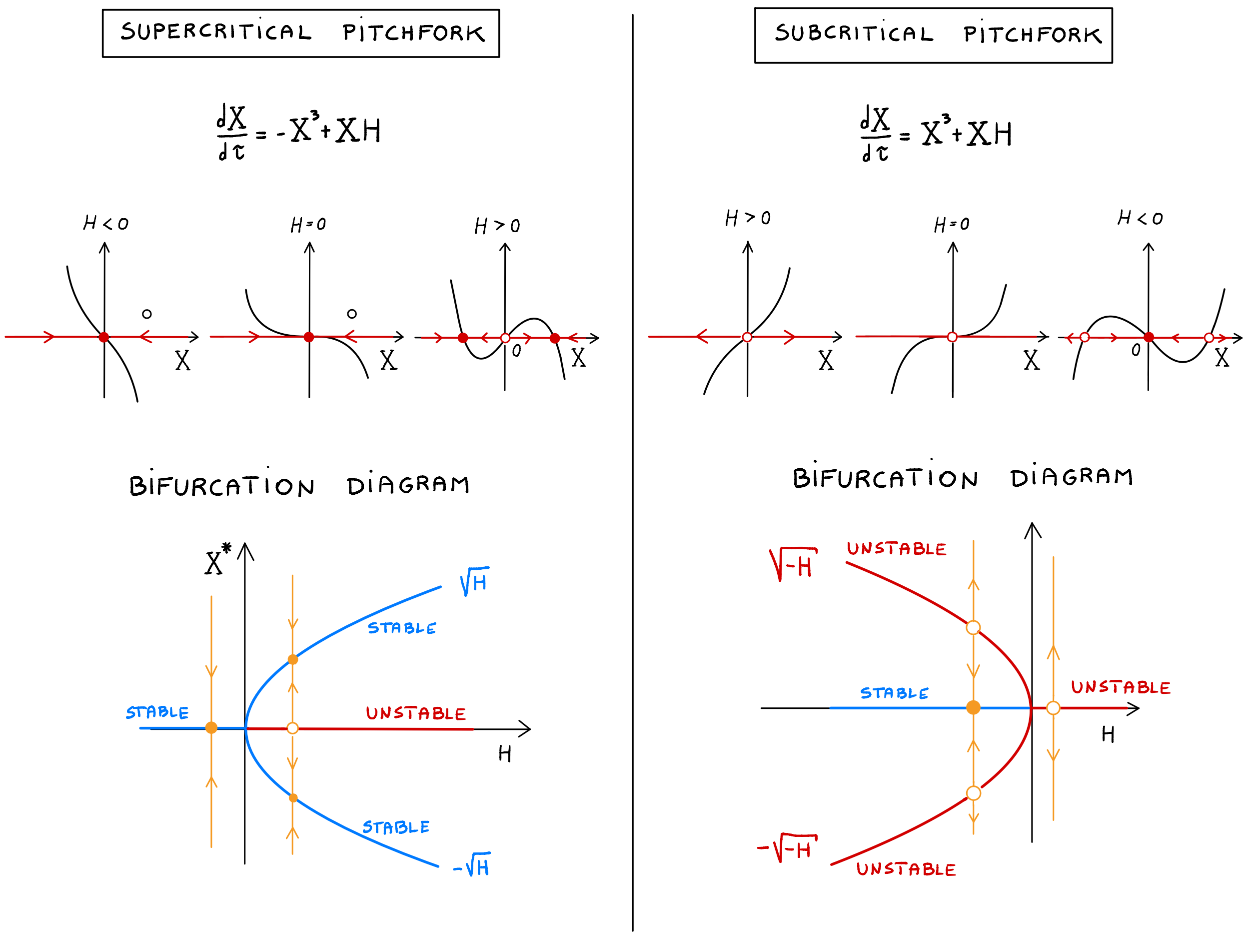

Figure 12: Bifurcation diagrams for supercritical and subcritical pitchfork bifurcations.

where we set

$$D = \frac{1}{6}\frac{\partial^3 f(x^*,\mu^*)}{\partial x^3} \neq 0, \qquad \text{and} \qquad C = \frac{\partial^2 f(x^*,\mu^*)}{\partial x \partial \mu} \neq 0. \tag{49}$$

Hence, to leading order, we obtain the following polynomial approximation of (2) at a pitchfork bifurcation point

$$\frac{dX}{dt} = DX^3 + CXR. \tag{50}$$

Dividing by the modulus of D yields

$$\frac{dX}{d\tau} = X^3 + XH \qquad (D > 0) \qquad \text{subcritical pitchfork,} \tag{51}$$

$$\frac{dX}{d\tau} = -X^3 + XH \qquad (D < 0) \qquad \text{supercritical pitchfork,} \tag{52}$$

where we set $\tau = |D|t$ and $H = CR/|D|$. Any dynamical system of the form (2) that undergoes a pitchfork bifurcation at (x^*, μ^*) can be written (to leading order in X and R after appropriate rescaling) either as (51) or (52) in a neighborhood of (x^*, μ^*). For this reason, (51) and (52) are referred to as the normal forms of the supercritical and subcritical pitchfork bifurcations, respectively. In Figure 12 plot the velocity vectors associated with the normal forms (51) or (52) and sketch the bifurcation diagrams.

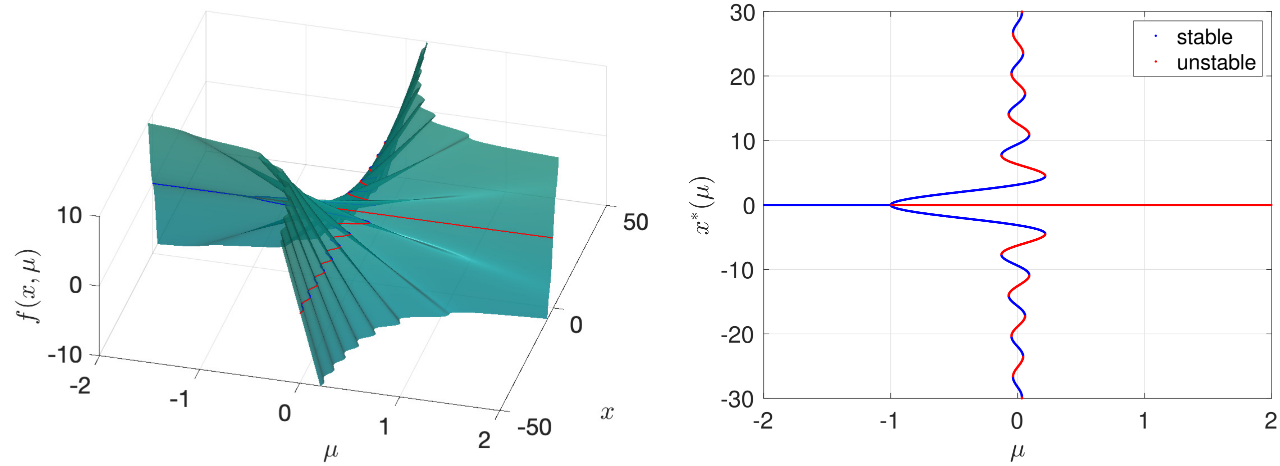

Figure 13: Bifurcation diagram for the system (53).

Example: Consider the system

$$\frac{dx}{dt} = \underbrace{\sin(x) + \mu x}_{f(x,\mu)}. \tag{53}$$

The fixed points are solutions to the transcendental equation

$$\sin(x) + \mu x = 0. \tag{54}$$

Clearly $x^* = 0$ is a fixed point for all μ. Note also that for $x \neq 0$ the bifurcation diagram is completely defined by the equation

$$\mu(x^*) = \frac{\sin(x^*)}{x^*}, \tag{55}$$

which explicitly expresses the bifurcation parameter as a function of the location of the fixed points. The derivative of the right hand side of (53) is

$$\frac{\partial f}{\partial x} = \cos(x) + \mu. \tag{56}$$

Hence, for $x^* = 0$ we have that $\mu^* = -1$ makes (56) equal to zero. This means that $(x^*, \mu^*) = (0, -1)$ could be a bifurcation point as it satisfies the necessary conditions (10). Let us verify that $(x^*, \mu^*) = (0, -1)$ is indeed a pitchfork bifurcation point. To this end, it is necessary and sufficient to verify the conditions in Theorem 4. We have

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x) \quad \Rightarrow \frac{\partial^2 f(x^*, \mu^*)}{\partial x^2} = 0, \tag{57}$$

$$\frac{\partial^3 f}{\partial x^3} = -\cos(x) \quad \Rightarrow \frac{\partial^3 f(x^*, \mu^*)}{\partial x^3} = -1, \tag{58}$$

$$\frac{\partial^2 f}{\partial x \partial \mu} = 1 \quad \Rightarrow \frac{\partial^2 f(x^*, \mu^*)}{\partial x \partial \mu} = 1. \tag{59}$$

Hence, there is a *supercritical pitchfork bifurcation* point at $(x^*, \mu^*) = (0, -1)$. Note, in fact, that (58) implies that the coefficient D in (49) is negative, and therefore the normal form representing the bifurcation in this case is (52). As shown in Figure 13, the system exhibits also an infinite number of saddle-node bifurcations as the parameter μ is varied. Such saddle-node bifurcations are defined analytically by equation (55).

Other bifurcations of equilibria. The Taylor expansion (17) can (to leading order) reduce to an arbitrary polynomial in the variables X and R. This opens the possibility for more *exotic* bifurcations of equilibria, in which a single fixed point can split into four or more branches (see Figure 5), or where multiple stable and unstable branches intersect at a single point. Such behavior arises when higher-degree terms dominate the local structure of the zero level set.