Exercise 1 Plot the phase portrait (including the nullclines) and classify the fixed points of the following linear dynamical systems

\begin{align*}
\dot{x}_1 &= x_1 + x_2 & \dot{x}_2 &= x_1 - 5x_2 \\
\dot{x}_1 &= x_1 + 4x_2 & \dot{x}_2 &= -x_1 - x_2 \\
\dot{x}_1 &= x_1 + x_2 & \dot{x}_2 &= -3x_1 - 2x_2 \\
\dot{x}_1 &= x_1 - x_2 & \dot{x}_2 &= -x_1 + x_2
\end{align*}

(1) (2) (3) (4)

If the eigenvectors are real, include them in your sketch.

Exercise 2 Compute the analytical solution to the systems (1) and (2) for an arbitrary initial condition \((x_1(0), x_2(0)) = (x_{01}, x_{02})\). If the eigenvalues are complex conjugate, express the solution in terms of real-valued functions only (Hint: use the Euler formula \(e^{ia} = \cos(a) + i\sin(a)\)).

Exercise 3 Show that any matrix of the form

\[ A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \]  \hspace{1cm} (5)

with \(a, b \neq 0\) has only a one-dimensional eigenspace. Also, determine the general solution to the system

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]  \hspace{1cm} (6)

and plot the phase portrait.

Exercise 4 The motion of a damped harmonic oscillator is described by the equation

\[ m\ddot{x} + b\dot{x} + kx = 0, \]  \hspace{1cm} (7)

where \(b \geq 0\) is the damping constant.

1. Rewrite equation (7) as a two-dimensional linear system;

2. Classify the fixed point at the origin, plot the nullclines, and sketch the phase portrait for all cases that can occur depending on the relative sizes of the parameters \(m, b\) and \(k\).