Lecture 3: Roots of complex polynomials

To characterize the roots of complex polynomials we first study the roots of a complex number.

Roots of a complex number. Let $z, w \in \mathbb{C}$ be two complex numbers, and $n \in \mathbb{N}$ a natural number. We have seen how to compute the n-th power of z (or w) using De Moivre's formula (see Lecture 2), i.e.,

$$z = |z|e^{i\vartheta} \quad \Rightarrow \quad z^n = |z|^n e^{in\vartheta}. \tag{1}$$

Now we consider the inverse operation, i.e., how to compute the n-th root of a complex number. We say that z is the n-th root[1] of w if

$$z^n = w. \tag{2}$$

This is the simplest polynomial equation involving complex numbers: here $w \in \mathbb{C}$ is *given* while $z \in \mathbb{C}$ is *to be determined*. We shall see hereafter that the polynomial equation (2) has exactly n solutions in $\mathbb{C}$. To compute such solutions it is convenient to first write both z and w in a polar form as

$$w = |w|e^{it} \qquad \text{and} \qquad z = |z|e^{i\vartheta}. \tag{3}$$

Taking the n-th power of z as in (1) and substituting it into (2) yields

$$|z|^n e^{in\theta} = |w|e^{it} \tag{4}$$

This equation is equivalent to the following system of equations

$$|z|^n = |w|, \qquad e^{in\vartheta} = e^{it}. \tag{5}$$

The first one admits the unique solution[2]

$$|z| = \sqrt[n]{|w|}. \tag{6}$$

The second equation $e^{in\vartheta} = e^{it}$ is an equality between two vectors on the unit circle in the complex plane, and it has exactly n distinct solutions $\{\vartheta_0, \ldots, \vartheta_{n-1}\}$. To compute such solutions we simply rewrite the equality $e^{in\vartheta} = e^{it}$ in a trigonometric form as

$$\cos(n\vartheta) = \cos(t), \qquad \sin(n\vartheta) = \sin(t). \tag{7}$$

This is a system of two *nonlinear* equations in the unknown variable ϑ.

How do we solve the nonlinear system (7) for ϑ? How many distinct solutions does it have?

- Clearly

  $$n\vartheta = t. \tag{8}$$

  is a solution to the system (7) since it satisfies both equations. In fact, substituting $n\vartheta = t$ into (7) yields two identities: $\cos(t) = \cos(t)$ and $\sin(t) = \sin(t)$. However, (8) is not the only solution. In fact, by using the periodicity of the cosine and sine functions we have that

  $$\cos(t + 2k\pi) = \cos(t) \qquad \sin(t + 2k\pi) = \sin(t) \quad \text{for all } k \in \mathbb{Z} \tag{9}$$

[1] In equation (2) w is the n-th power of z while z is the n-th root of w.

[2] Recall that $|z| \geq 0$ and $|w| \geq 0$. Therefore there exists a unique solution to $|z|^n = |w|$.

This means that

$$n\vartheta = t + 2k\pi \qquad k \in \mathbb{Z} \tag{10}$$

are solutions. These solutions however are not all distinct. In fact, we have seen that $\vartheta \pm 2\pi$ identifies the same complex number on the unit circle. Therefore the only distinct solutions of (7) are

$$n\vartheta_k = t + 2k\pi \qquad k = 0, \ldots, n-1. \tag{11}$$

Therefore, the complex n-the roots of a number $w \in \mathbb{C}$ can be written explicitly as follows:

$$z^n = w \qquad \Leftrightarrow \qquad \boxed{z_k = \sqrt[n]{|w|}e^{i\vartheta_k} \qquad \vartheta_k = \frac{t + 2k\pi}{n} \qquad k = 0, \ldots, n-1} \tag{12}$$

where $|w|$ and t are, respectively, the modulus and the argument of the complex number w. Note that all complex roots of a number w lie on a circle with radius $\sqrt[n]{|w|}$ in the complex plane.

Example: Compute the complex 4-th roots the real number $w = -1$. Such roots are defined by the (complex) solutions to the equation

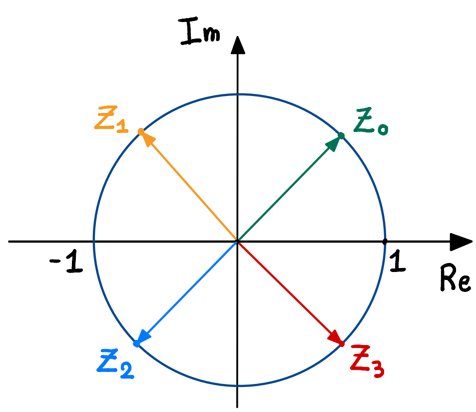
$$z^4 = -1. \tag{13}$$

By applying formula (12) we immediately get

$$z_k = e^{i(\pi + 2k\pi)/4} \qquad k = 0, 1, 2, 3.$$

In fact, the modulus of w is equal to 1, and therefore $\sqrt[4]{|w|} = 1$. The four complex 4-th roots of -1 can be written in an algebraic form as follows

$$z_0 = \frac{1+i}{\sqrt{2}}, \qquad z_1 = \frac{-1+i}{\sqrt{2}}, \qquad z_2 = \frac{-1-i}{\sqrt{2}}, \qquad z_3 = \frac{1-i}{\sqrt{2}}.$$



It can be verified that each z_k in (14) satisfies indeed $z_k^4 = -1$. For example, using the algebraic form we have

$$\begin{aligned} z_0^4 &= \frac{(1+i)^4}{4} = \frac{(1+i)^2(1+i)^2}{4} = \frac{(2i)^2}{4} = -1, \\ z_1^4 &= \frac{(-1+i)^4}{4} = \frac{(-1+i)^2(-1+i)^2}{4} = \frac{(-2i)^2}{4} = -1. \end{aligned} \tag{14}$$

Example: Compute the complex cubic roots of the real number $w = 2$. The cubic roots are complex solutions of the polynomial equation

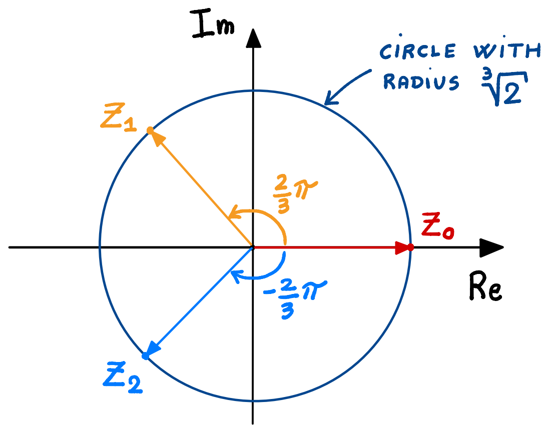
$$z^3 = 2 \tag{15}$$

By using (12) we immediately obtain

$$z_k = \sqrt[3]{2} e^{i2k\pi/3} \qquad k = 0, 1, 2$$

i.e.,

$$z_0 = \sqrt[3]{2}, \qquad z_1 = \frac{\sqrt[3]{2}}{2}\left(-1 + i\sqrt{3}\right), \qquad z_2 = \frac{\sqrt[3]{2}}{2}\left(-1 - i\sqrt{3}\right).$$



We remark that if we solve $z^3 = 1$ in $\mathbb{R}$ instead of $\mathbb{C}$ then we obtain a unique solution, i.e., $z = 1$. On the other hand in $\mathbb{C}$ we have three solutions: one real, and two *complex conjugates*.

Remark: Formula (12) suggests that once the first n-th root z_0 is found, then all others can be obtained by simply dividing the circle with radius $|z| = \sqrt[n]{|w|}$ into n evenly-spaced parts!

Roots of quadratic polynomial equations in $\mathbb{C}$. Consider the following quadratic polynomial[3]

$$az^2 + bz + c = 0, \tag{17}$$

where a, b, and c can be complex numbers. Divide (17) by a

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0$$

and complete the square

$$\left(z + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0. \tag{18}$$

Upon definition of

$$\delta = z + \frac{b}{2a} \tag{19}$$

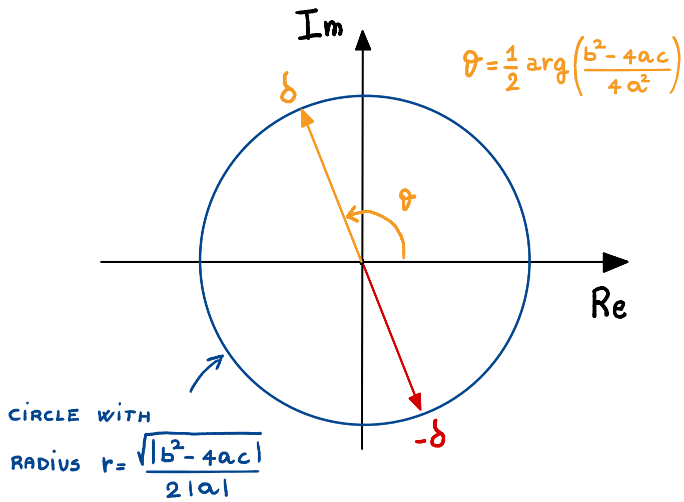
[3]An example of a quadratic polynomial with complex coefficients is

$$(1 + i)z^2 + 5z - i = 0. \tag{16}$$

we can write (18) as

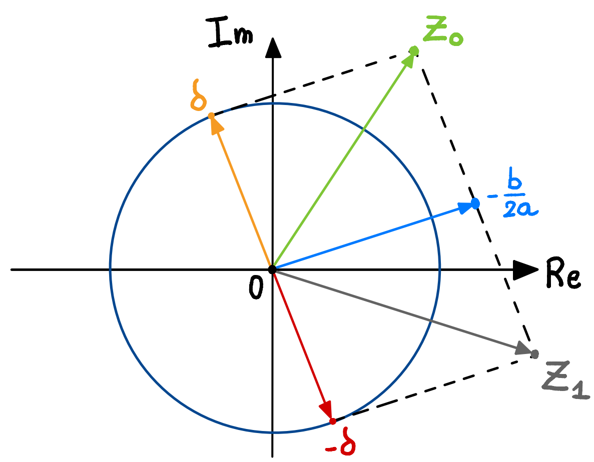
$$\delta^2 = \frac{b^2 - 4ac}{4a^2}.$$

This equation can be solved by taking the square root of the complex number $b^2 - 4ac/4a^2$. As is well-known, this yields two complex numbers δ and $-\delta$ opposite to each other and sitting on a circle with radius $|b^2 - 4ac|^{1/2}/(2|a|)$.



By using (19) we see that the solution of the quadratic polynomial equation (17) is then

$$z_0 = \delta - \frac{b}{2a}, \qquad z_1 = -\delta - \frac{b}{2a}. \tag{20}$$



Example: Consider the polynomial equation

$$z^2 + z + 1 = 0. \tag{21}$$

This equation has real coefficients but no solution in $\mathbb{R}$. By using the mathematical steps discussed above it can be shown that (21) can be written as

$$\delta^2 = -\frac{3}{4} \qquad \text{where} \qquad \delta = z + \frac{1}{2}. \tag{22}$$

Therefore the two complex solutions are

$$z_{0,1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \tag{23}$$

Quadratic polynomials with real coefficients have roots that are either real or complex conjugates. The roots can also be computed with the standard quadratic formula in this case.

Example: Consider the polynomial equation

$$z^2 + iz + 1 + i = 0. \tag{24}$$

By using the mathematical steps discussed above it can be shown that (24) can be written as

$$\delta = z + \frac{i}{2} \qquad \delta^2 = -\frac{5}{4} - i. \tag{25}$$

The polar form of the complex number $-5/4 - i$ is

$$-\frac{5}{4} - i = \frac{\sqrt{41}}{4} e^{i(\arctan(4/5)+\pi)}. \tag{26}$$

Therefore, the two solutions of $\delta^2 = -5/4 - i$ are

$$\delta_0 = \frac{\sqrt[4]{41}}{2} e^{i(\arctan(4/5)+\pi)/2}, \qquad \delta_1 = \frac{\sqrt[4]{41}}{2} e^{i(\arctan(4/5)+3\pi)/2}. \tag{27}$$

This implies that the roots of (24) are

$$z_0 = \frac{1}{2}\left(-i + \sqrt[4]{41} e^{i(\arctan(4/5)+\pi)/2}\right), \qquad z_1 = \frac{1}{2}\left(-i + \sqrt[4]{41} e^{i(\arctan(4/5)+3\pi)/2}\right). \tag{28}$$

Roots of complex polynomials. In the previous section we have seen how to compute the roots of quadratic polynomials with complex coefficients. A natural question is whether it is possible to generalize such computations to complex polynomials of degree $n > 2$. These polynomials can be written as

$$p(z) = a_n z^n + \cdots + a_1 z + a_0 \qquad a_i \in \mathbb{C}. \tag{29}$$

An even deeper question is whether polynomials of the form (29) actually have roots. This question was answered in 1799 by Gauss.

Theorem 1 (Fundamental theorem of algebra, Gauss 1799)**.** Every non-constant polynomial of the form (29) has at least one complex root.

By applying Gauss's theorem recursively it is straightforward to conclude that (29) has exactly n complex roots. This is summarized in the following Corollary.

Corollary 1. Every non-constant polynomial of the form (29) has exactly n complex roots.

Proof. Let $z_1 \in \mathbb{C}$ be a root of (29). We known that such a root exists because of Theorem 1. Let us first transform (29) to a monic polynomial (just divide by $a_n \neq 0$)

$$\hat{p}(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0, \qquad b_i = \frac{a_i}{a_n} \quad i = 0, \ldots, n-1. \tag{30}$$

Obviously we can factor out z_1 as

$$\hat{p}(z) = (z - z_1)\hat{p}_1(z) \tag{31}$$

where $\hat{p}_1(z)$ is a polynomial of degree $n-1$ obtained by dividing $\hat{p}(x)$ by $(z - z_1)$. The reminder of such polynomial division is zero because z_1 is a root of $\hat{p}(z)$. At this point we apply Theorem 1 again to $\hat{p}_1(x)$ to conclude that there exists another root z_2 and a polynomial $\hat{p}_2(x)$ of degree $n-2$ such that

$$\hat{p}(z) = (z - z_1)(z - z_2)\hat{p}_2(z). \tag{32}$$

Proceeding recursively we conclude that the polynomial (29) can be factorized as

$$\hat{p}(z) = (z - z_1)(z - z_2)\cdots(z - z_n). \tag{33}$$

This means that $\hat{p}(z)$ has exactly n roots in $\mathbb{C}$ (not necessarily distinct).

□

Regarding the computation of the roots, Ruffini (1799) and Abel (1824) proved that it is impossible to obtain closed form expressions for the roots of arbitrary polynomials of degree $n \geq 5$. This means that if we are interested in computing the roots of a given polynomial with degree $n \geq 5$ then we have to proceed numerically.

Remark: Effective algorithms to compute the roots of (29) are based on eigenvalue solvers. In fact, it can be shown that eigenvalues of the following *companion matrix*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & 1 & -b_{n-1} \end{bmatrix} \tag{34}$$

coincide with the roots of the polynomials (30) and (29).

The following theorem characterizes the roots of polynomials with *real* coefficients.

Theorem 2. Let $z_0 \in \mathbb{C}$ be a root of a polynomial with real coefficients. Then z_0^* (complex conjugate of z_0) is also a root.

Proof. Let

$$p(z) = \sum_{k=0}^{n} a_k z^k \tag{35}$$

be a polynomial with real coefficients $\{a_n, \ldots, a_0\}$. If z_0 is a root of $p(z)$ then

$$\sum_{k=0}^{n} a_k z_0^k = 0. \tag{36}$$

By taking the complex conjugate of (36) and recalling that[4]

$$(z_0^k)^* = (z_0^*)^k \qquad a_k^* = a_k \tag{37}$$

we obtain

$$\sum_{k=0}^{n} a_k (z_0^*)^k = 0. \tag{38}$$

Therefore if z_0 is a root of $p(z)$ then z_0^* is also a root of $p(z)$.

□

Theorem 2 states the roots of a polynomial of degree n with real coefficients are either real or complex conjugates. This implies that the number of complex roots is always even for polynomials with real coefficients.

[4]Equation (37) follows from $(z_0^2)^* = (z_0 z_0)^* = z_0^* z_0^* = (z_0^*)^2$.