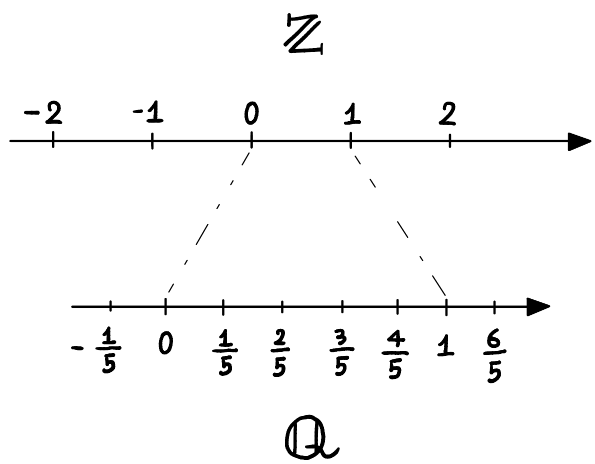
Lecture 1: Real numbers

In this lecture we briefly discuss the set of real numbers $\mathbb{R}$, and show how such set can be constructed based on successive extensions of the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

The main steps are:

1. Construct $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$ (integer numbers).
2. Construct $\mathbb{Q} = \left\{ \frac{p}{q} \ : \ p, q \in \mathbb{Z}, \quad q \neq 0 \right\}$ (rational numbers).
3. Define the set of irrational numbers and add it to the set of rational numbers to obtain the set of real numbers.



Of course we can go on and look for four more rational numbers between 0 and 1/5, i.e.,

$$\left\{ \frac{1}{25}, \frac{2}{25}, \frac{3}{25}, \frac{4}{25} \right\}, \tag{1}$$

etc. Clearly, we have[1]

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}. \tag{2}$$

Rational numbers either have a finite number of decimal digits or an infinite number of decimal digits repeating periodically. For example:

$$\begin{aligned}
\frac{3}{4} =& 0.75 \qquad \text{(finite number of decimals)},\\
\frac{1}{3} =& 0.333333\ldots = 0.\overline{3} \qquad \text{(infinite decimals repeating periodically)},\\
\frac{1}{7} =& 0.\underbrace{142857}\underbrace{142857}\ldots = 0.\overline{142857} \qquad \text{(infinite decimals repeating periodically)}.
\end{aligned}$$

[1]In mathematics, the symbol "$\subset$" means "subset of". Note that $\mathbb{N}$ is a subset of $\mathbb{Z}$ because $\mathbb{Z}$ includes all integer numbers $\{1, 2, 3, \ldots\}$. In addition, $\mathbb{Z}$ includes the negative of all integer numbers and the element zero $\{0\}$. Of course, the set of rational numbers $\mathbb{Q}$ includes, by definition, the set of natural numbers as well as the set of integer numbers.

Moreover, the sum or the products of two rational numbers is still a rational number. For example,

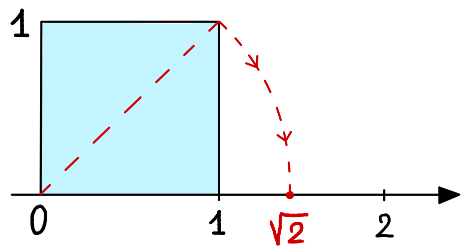
$$\frac{1}{3} + \frac{2}{15} = \frac{7}{15}, \qquad \frac{1}{3} \times \frac{2}{15} = \frac{2}{45}. \tag{3}$$

You may have heard that the set of rational numbers is not "complete". In other words, there are numbers on the (continuum) horizontal lines sketched above that cannot be represented as a ratio between two integers, i.e., as a rational number. One of such numbers is the square root of 2

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718 7\cdots \tag{4}$$

which has an infinite number of decimals that do not repeat periodically as in the case of rational numbers.

The number $\sqrt{2}$ is indeed an *irrational* number that can be visualized by rotating the diagonal of a unit square by 45 degrees clockwise as follows



Recall, in fact, that by the Pythagorean theorem, the length of the diagonal of the unit square is

$$\sqrt{1^2 + 1^2} = \sqrt{2}. \tag{5}$$

Remarkably, no matter how hard we try to cover the continuum line with rational numbers we find out that the number of "holes" left to be filled is *infinite* and *uncountable* (cardinality larger than $\mathbb{N}$). Hereafter we rigorously show that $\sqrt{2}$ is indeed not a rational number. To this end, we formulate the following Theorem[2]:

Theorem. There is no rational number the square of which equals 2. Equivalently, there is no rational number that equals $\sqrt{2}$.

Proof. Suppose that there exists a rational number p/q (irreducible fraction[3]) the square of which equals two:

$$\left(\frac{p}{q}\right)^2 = 2 \quad \Rightarrow \quad p^2 = 2q^2. \tag{6}$$

Clearly, $p^2 = 2q^2$ is an even natural number (2 times the natural number q^2 is necessarily even). This implies that p is an even integer number, and therefore can be written as

$$p = 2s \qquad \text{for some } s \in \mathbb{Z}. \tag{7}$$

[2] A Theorem is a statement that is not self-evident but can be proved (or disproved) by a sequence of logical or mathematical operations.

[3] An irreducible fraction is a fraction that cannot be simplified any further. For example, $3/2$ is an irreducible fraction while $6/4$ is not an irreducible fraction as both the numerator and the denominator can be divided by 2 to obtain $3/2$.

Substituting equation (7) into (6) yields

$$\frac{2^2 s^2}{q^2} = 2 \quad \Rightarrow \quad q^2 = 2s^2 \quad \Rightarrow \quad q \quad \text{is an even number.} \tag{8}$$

But this contradicts the fact that p/q is an irreducible fraction. Indeed, we just concluded that both p and q are divisible by 2 since they are both even numbers. In summary, the assumption that there exists a rational number p/q equals to $\sqrt{2}$ yields a contradiction, and therefore the assumption must be wrong, meaning that such a rational number cannot exist.

□

The technique we just used to prove the Theorem above is known as "proof by contradiction" ("Reductio ad absurdum" in Latin). Essentially it is a form of argument that establishes a statement by arriving at a contradiction or at something that is impossible or absurd, even when the initial assumption is the negation of the statement to be proved. For example, let us prove the following statement

"There isn't a smallest positive rational number"

by using the proof by contradiction technique. To this end, we first assume that there is a smallest positive rational number (negation of the statement) and immediately notice that dividing such rational number by 2 (or any other integer number larger than 2) yields another rational number that is smaller than the one we started with. This contradicts the hypothesis that there is a smallest rational number. Therefore the statement "There isn't a smallest rational number" must be true.

There are many other examples of numbers that cannot be represented as a ratio between two integer numbers (i.e., rationals). Such numbers are called *irrational numbers*, and they have an infinite number of decimals (non-repeating). Moreover, there are infinite irrational numbers (uncountably many!). Well known examples of irrational numbers are: $\pi = 3.141592653\cdots$, $\sqrt{2} = 1.41421356\cdots$, $\sqrt{3} = 1.73205080\cdots$, $e = 2.71828182845\cdots$ (Napier number).

Remarkably, all irrational numbers can be obtained as limits of suitable sequences of rational numbers. For example,

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) \qquad \text{(Leibnitz sequence)}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

Similarly, $\sqrt{2}$ can be obtained by iterating (an infinite number of times) the following sequence of rational numbers:

$$S_{n+1} = \frac{S_n}{2} + \frac{1}{S_n} \qquad S_0 = 1 \tag{9}$$

i.e.,

$$S_0 = 1, \qquad S_1 = \frac{3}{2} = 1.5, \qquad S_2 = \frac{17}{12} = 1.41\overline{6}, \qquad S_3 = \frac{577}{408} = 1.414\overline{2156862745098039}, \quad \ldots$$

The sequences above are not unique, meaning that there are many other sequences of rationals converging to the same irrationals. For example,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad \text{(Bernoulli sequence)}.$$

In any case, such sequences do exist, and they can represent (in their limit) all irrational numbers[4]. By adding (formally) the set of irrational numbers (denoted by $\mathbb{I}$), which is uncountable, to the set of the rationals $\mathbb{Q}$ we obtain the set of *real numbers*[5]:

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I} \tag{10}$$

A remarkable result of number theory says that both $\mathbb{Q}$ and $\mathbb{I}$ are "dense" in $\mathbb{R}$. This means that any real number can be obtained as limit point of sequences of rational numbers or sequences of irrational numbers. Moreover, between any two distinct real numbers there always exists a rational number and an irrational one.

Axiomatic definition of $\mathbb{R}$

The set of real numbers can be defined in an axiomatic way. An axiom is statement or a proposition which is regarded as being established, accepted, or self-evidently true. Hence, by defining $\mathbb{R}$ in terms of axioms we specify properties of $\mathbb{R}$ that are self-evidently true.

Field axioms. The set of real number $\mathbb{R}$ is a (algebraic) field, i.e., it is a set in which we can define two operations (addition "+" and multiplication[6]) with the following properties:

1. Associative property[7]:

$$\forall x, y, z \in \mathbb{R}, \quad (x+y)+z = x+(y+z) \quad \text{and} \quad (xy)z = x(yz).$$

2. Commutative property:

$$\forall x, y \in \mathbb{R}, \ x+y = y+x \quad \text{and} \quad xy = yx.$$

3. Additive neutral element:

There exists an element of $\mathbb{R}$, denoted by 0, such that $\forall x \in \mathbb{R}$, $x + 0 = x$.

4. Multiplicative neutral element:

There exists an element of $\mathbb{R}$, denoted by 1, such that $\forall x \in \mathbb{R}$, $1x = x$.

[4] The sequences are actually used in practice to compute approximations of irrational numbers. For example, in 2016 Ron Watkins used the sequence (9) to compute 10 trillion digits of $\sqrt{2}$ (see http://www.numberworld.org/digits/Sqrt(2)).

[5] Note that $\mathbb{R}$ can also be thought of as $\mathbb{Q}$ plus all limit points of converging sequences of rational numbers.

[6] The multiplication operation between two elements $x, y \in \mathbb{R}$ is denoted simply as xy.

[7] In mathematics, the symbol $\forall$ means "for all", while the symbol $\in$ means "in". Hence, writing $\forall x, y, z \in \mathbb{R}$ can be spelled out as follows: "for all x, y and z in the set of real numbers".

5. Inverse with respect to addition:

$$\forall x \in \mathbb{R}, \text{ there exists } y \in \mathbb{R} \text{ such that } x + y = 0\,(y = -x) \quad (y \text{ is the opposite of } x).$$

6. Inverse with respect to multiplication:

$$\forall x \in \mathbb{R} \setminus \{0\}, \text{ there exists } y \in \mathbb{R}|\ x \cdot y = 1 \quad (y \text{ is the inverse of } x).$$

7. Distributive property of multiplication:

$$\forall x, y, z \in \mathbb{R} \quad x(y + z) = xy + xz.$$

Remark: The set $\mathbb{R}$ is closed under addition and multiplication. This means that addition and product of real numbers is still a real number. In general, any set satisfying properties 1.-7. is a called a (algebraic) *field*. In particular, it can be verified that: 1) $\mathbb{N}$ is not a field; 2) $\mathbb{Z}$ is not a field (the inverse with respect to multiplication is not in $\mathbb{Z}$); 3) $\mathbb{Q}$ is a field; 4) $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ is not a field (the product of two irrationals can be an integer $\sqrt{2}\sqrt{2} = 2$).

By using the field axioms it is easy define subtraction and division between two real numbers as:

Subtraction: $\forall x, y \in \mathbb{R},\ x - y = x + (-y)$ (subtraction seen as adding the opposite of y).

Division: $\forall x, y \in \mathbb{R}$, $y \neq 0$, $x/y = xy^{-1}$ (division seen as multiplying by the inverse of y).

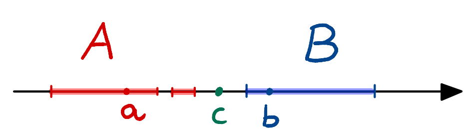
Ordering axioms. The field of real numbers $\mathbb{R}$ is totally ordered, i.e., we can define in $\mathbb{R}$ an ordering relation $\leq$ such that for all $x, y \in \mathbb{R}$:

1. $x \leq y \quad \Rightarrow \quad x + z \leq y + z, \qquad \forall z \in \mathbb{R}.$
2. $x \leq y \quad \Rightarrow \quad xz \leq yz, \qquad \forall z \geq 0.$

The mathematical symbol $\leq$ means "less or equal". Similarly, "$\geq$" means "greater or equal". So the ordering axiom number 2. can be phrased as follows: "Let x and y be two arbitrary real numbers; if x is smaller or equal than y, and z is any non-negative real number, then xz is smaller or equal than yz."

Remark: The ordering axioms say that all elements of $\mathbb{R}$ are ordered, i.e., we can always tell which element is bigger or smaller than any other element. That is why the lines sketched at Page 1 and Page 2 have one arrow (not two!) that indicates the direction in which the numbers are increasing.

Completeness axiom. $\mathbb{R}$ is a field that is totally ordered and complete. "Complete" means that for every subsets $A, B \subseteq \mathbb{R}$ not empty and separated (i.e., such that $a \leq b$ $\forall a \in A$ and $\forall b \in B$) there exists at least one $c \in \mathbb{R}$ such that $a \leq c \leq b$.

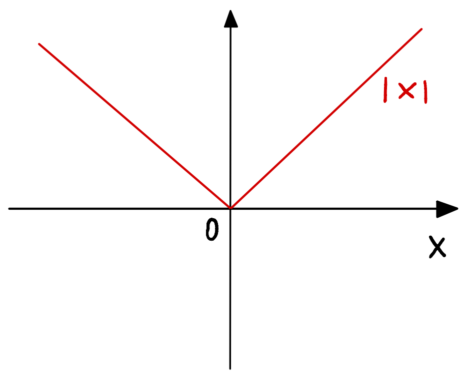


The completeness axiom assures that the set $\mathbb{R}$ has no "holes" in it. Also, it can be shown that between two real numbers there is always an irrational and a rational, and between two rational numbers there is always a real number and an irrational number.

Absolute value. The absolute value of a real number is a function defined as

$$
\begin{aligned}
|\cdot| &: \mathbb{R} \to \mathbb{R}^+ \\
x \to |x| &= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}
\end{aligned}
\tag{11}
$$

Here, $\mathbb{R}^+$ denotes the set of non-negative real numbers, i.e., $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.



The absolute value function satisfies a certain number of properties summarized in the following Theorem. Each of the properties can be proved based on the definition of $|\cdot|$.

Theorem (Properties of the absolute value)**.** Let $a, b \in \mathbb{R}$. Then we have

1. $|a| \geq 0$
2. $|a| = 0 \Leftrightarrow a = 0$
3. $|a| = |-a|$
4. $-|a| \leq a \leq |a|$
5. $|a|^2 = a^2$
6. $|ab| = |a||b|$
7. $|a + b| \leq |a| + |b|$ (Triangle inequality)
8. $|a + b| \geq ||a| - |b||$ (Reverse triangle inequality)

Proof. Let us prove the triangle inequality and the reverse triangle inequality. For every $a, b \in \mathbb{R}$ we have[8]

$$
a + b \leq |a| + |b| \quad \text{and} \quad a + b \geq -|a| - |b|. \tag{12}
$$

[8]To prove (12) we notice that for all $a, b \in \mathbb{R}$ we have $a \leq |a|$ and $b \leq |b|$. Therefore $a + b \leq |a| + |b|$. Similarly, we have $-|a| \leq a$ and $-|b| \leq b$, which imply that $-|a| - |b| \leq a + b$. Multiplying the last inequality by -1 yields $-(a + b) \leq |a| + |b|$.

This implies that

$$-\left(|a|+|b|\right) \leq a+b \leq |a|+|b| \tag{13}$$

i.e.,

$$|a+b| \leq |a|+|b| \quad \text{for all} \quad a,b \in \mathbb{R}. \tag{14}$$

The last step follows from the fact that if c is any non-negative real number then

$$|a| \leq c \quad \Leftrightarrow \quad -c \leq a \leq c. \tag{15}$$

Similarly, for the reverse triangle inequality we observe that

$$a = a+b-b \quad \Rightarrow |a| \leq |a+b| + |b|\,, \tag{16}$$

$$b = b+a-a \quad \Rightarrow |b| \leq |a+b| + |a|\,. \tag{17}$$

Therefore

$$-|a+b| \leq |a|-|b| \leq |a+b| \tag{18}$$

i.e.,

$$\left||a|-|b|\right| \leq |a+b|\,. \tag{19}$$

$\square$

By using the definition of absolute value we can define closed and open intervals of the real line with endpoint a and b ($a,b\in\mathbb{R}$, $a<b$) as

$$[a,b] = \{x\in\mathbb{R}|\ a\leq x\leq b\} = \left\{x\in\mathbb{R} : \left|x-\frac{a+b}{2}\right| \leq \frac{b-a}{2}\right\} \quad \text{(closed interval)}, \tag{20}$$

$$]a,b[= \{x\in\mathbb{R}|\ a< x< b\} = \left\{x\in\mathbb{R} : \left|x-\frac{a+b}{2}\right| < \frac{b-a}{2}\right\} \quad \text{(open interval)}. \tag{21}$$

Solution to linear and nonlinear equations

It is good practice to specify in which space we are looking for solutions of a certain equation. For instance, the following linear equation

$$2x = 1 \tag{22}$$

has no solution in $\mathbb{N}$ and no solution in $\mathbb{Z}$, but it has a unique solution in $\mathbb{Q}$ equal to $x = 1/2$, and of course a unique solution in $\mathbb{R}$ (since $\mathbb{Q}\subset\mathbb{R}$). Many nonlinear equations, however, do not admit a solution in $\mathbb{R}$. For example, the following polynomial (quadratic) equation

$$x^2 + 1 = 0 \tag{23}$$

has no solution in $\mathbb{R}$. In fact, the square of any real number x is non-negative, i.e., $x^2 \geq 0$ for all $x\in\mathbb{R}$. Hence, there is no element in $\mathbb{R}$ such that $x^2 = -1$, and therefore (23) has no solution in $\mathbb{R}$.

If we are interested in defining a solutions to equation (23), then we need to utilize a different set of numbers. In particular, such a set should include particular type of numbers the square of which is negative and real. As we will see such numbers are called *imaginary numbers*, and will be described in detail in the next lecture. Imaginary numbers are a subset of a more general set of numbers which is *complex numbers* and denoted as $\mathbb{C}$. Complex numbers were historically developed to make sense of solutions of polynomial equations (i.e., zeros of polynomials). For instance it was shown that:

Theorem (Fundamental theorem of algebra)**.** Every non-constant polynomial of the form

$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{24}$$

with real or complex coefficients $\{a_n, \ldots, a_0\}$ has at least one complex root.

By applying this theorem recursively, it can be shown that every polynomial of degree n has exactly n complex roots (which may not be all distinct).

Complex numbers and complex functions pay a fundamental role in a variety of applications, e.g., series expansions of periodic functions, signal processing, solution to PDEs via Fourier series/transforms, quantum mechanics (e.g., Schrödinger equation), fluid dynamics (e.g., Joukowsky transformations for airfoil design), conformal maps, nonlinear dynamics and control, image processing, wave propagation, etc.