Lecture 2: Complex numbers

The quadratic equation

$$z^2 + 1 = 0 \tag{1}$$

has no solution in $\mathbb{R}$. In fact, there is no real number such that $z^2 = -1$ (recall that the square of any real number is either positive or equal to zero). We can still define solutions to equation (1), but we have to seek them in a different set of numbers. In particular, such a set must include new types of numbers the square of which is negative and real. These numbers are called *imaginary numbers.*

Let "i" one of such numbers, i.e., an imaginary number defined as $i^2 = -1$, or equivalently

$$i = \sqrt{-1}. \tag{2}$$

Clearly, $z = i$ is a solution of equation (1). In fact,

$$z^2 + 1 = i^2 + 1 = -1 + 1 = 0. \tag{3}$$

Moreover, $z = -i$ is another solution of (1) since

$$z^2 + 1 = (-i)^2 + 1 = (-1)^2 i^2 + 1 = -1 + 1 = 0. \tag{4}$$

Next, consider the polynomial equation

$$z^2 + z + 1 = 0 \tag{5}$$

You certainly know that the two roots of this equation can be computed based on the formula

$$z_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}. \tag{6}$$

Although the number within the square root is negative (the parabola $y = z^2 + 2z + 1$ has no intersection with the axis $y = 0$) we can write (6) as

$$z_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \tag{7}$$

This suggests that the new set of numbers we are interested in is of the form

$$z = x + iy \qquad \text{(complex number)} \tag{8}$$

where x and y are real numbers and i is the imaginary unit defined in (2). Specifically, x is called the *real part* of z, and y is called the *imaginary part* of z.

Numbers of the form (8) are called *complex numbers.* The set of all complex numbers will be denoted by

$$\mathbb{C} = \{z = x + iy : \, x, y \in \mathbb{R}\}. \tag{9}$$

As we shall see hereafter, $\mathbb{C}$ is an algebraic field, i.e., is possible to define in $\mathbb{C}$ addition and multiplication operations satisfying the same axioms of we have seen in Lecture 1 for $\mathbb{R}$ (field axioms).

Addition and Multiplication in $\mathbb{C}$. Consider the following complex numbers

$$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2. \tag{10}$$

It is natural to define addition and multiplication in $\mathbb{C}$ by using the addition and multiplication operations we defined in $\mathbb{R}$ (see Lecture 1). Specifically, we define

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \qquad \text{(addition operation)} \tag{11}$$

Note that $z_1 + z_2$ is still of the form $x + iy$ (with $x = (x_1 + x_2)$ and $y = (y_1 + y_2)$), and therefore $\mathbb{C}$ is closed[1] under the addition operation "+" defined in (11). Similarly,

$$\begin{aligned} z_1 z_2 =& (x_1 + iy_1)(x_2 + iy_2) \\ =& (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \qquad \text{(multiplication operation)} \end{aligned} \tag{12}$$

Again, z_1z_2 is a complex number, i.e., a number of the form $x + iy$ (with $x = (x_1x_2 - y_1y_2)$ and $y = (x_1y_2 + x_2y_1)$). This means that $\mathbb{C}$ is closed under the multiplication operation defined in (12)

It is easy to show that the set of complex numbers $\mathbb{C}$, with the addition and multiplication operations defined in (11) and (12) is a field. In other words, for all $z_1, z_2, z_3 \in \mathbb{C}$ we have that:

1. Addition and multiplication are commutative

$$z_1 + z_2 = z_2 + z_1 \qquad z_1z_2 = z_2z_1 \tag{13}$$

2. Addition and multiplication in are associative

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \qquad (z_1z_2)z_3 = z_1(z_2z_3) \tag{14}$$

3. The distributive property of multiplication relative to addition holds

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \tag{15}$$

4. There exists neutral elements for both addition and multiplication

$$\begin{aligned} &z_1 + z_2 = z_1 \quad \Rightarrow \quad z_2 = 0 + i0 \qquad \text{(additive neutral element)} \\ &z_1z_2 = z_1 \qquad \Rightarrow \quad z_2 = 1 + i0 \qquad \text{(multiplicative neutral element)} \end{aligned} \tag{16}$$

Based on the definition of additive and multiplicative neutrals it is straightforward to define the *opposite* and the *inverse* of a complex number. To this end, let $z = x + y$

$$\begin{aligned} &z + z_1 = 0 + 0i \quad \Rightarrow \quad z_1 = -x - iy \qquad \text{(opposite of } z\text{)} \\ &zz_2 = 1 + 0i \qquad \Rightarrow \quad z_2 = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \qquad \text{(inverse of } z\text{)} \end{aligned} \tag{17}$$

Let us denote z_1 as $-z$ and z_2 as $1/z$. Note that equations (17) allow us to define *subtraction* and *division* between two complex numbers in terms of addition and multiplication. In fact, subtraction

[1] We say that $\mathbb{C}$ is closed under the addition operation + defined in (11) if for all $z_1, z_2 \in \mathbb{C}$ we have that $(z_1 + z_2) \in \mathbb{C}$.

of z_2 from z_1 is the same as adding the opposite of z_2 to z_1. Similarly, z_1/z_2 is the same as multiplying z_1 by the inverse of z_2.

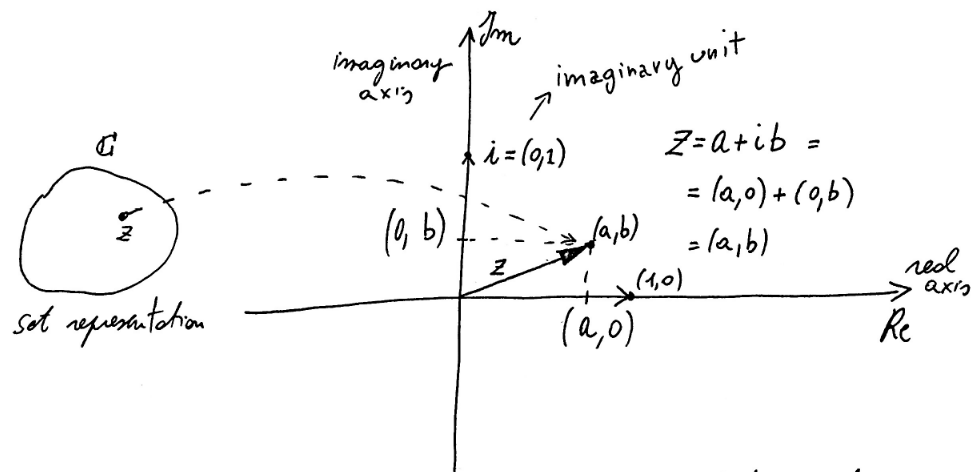
Remark: We have now set up all the machinery to perform any type of algebraic calculation between complex numbers, including addition, subtraction, multiplication and division.

Remark: From what has been said, it is clear that $\mathbb{C}$ includes $\mathbb{R}$ as a subset, i.e., $\mathbb{R} \subset \mathbb{C}$. This can be seen by noting that real numbers are simply complex numbers with zero imaginary part. Moreover, the addition and multiplication operations we defined in $\mathbb{C}$, i.e., equations (11)-(12), reduce to addition and multiplication between real numbers if we set to zero the imaginary parts. Hence $\mathbb{R} \subset \mathbb{C}$.

Remark: $\mathbb{C}$ is not an ordered field. In other words, it does not make sense to write inequalities between complex numbers.

Graphical representation of complex numbers. Clearly, there is a one-to-one correspondence between the complex number $z = x+iy$ and the real numbers $x, y \in \mathbb{R}$. This means that z identifies uniquely x and y, and conversely (x, y) identifies uniquely the complex number z. This suggests that we could represent (x, y) as a point (or a vector) in the Cartesian plane.

When the Cartesian plane is used to represent complex numbers, it is usually called *complex plane*. In this setting, the x-axis is called *real axis* while the y-axis is called *imaginary axis*.



The complex plane us to easily visualize additions (parallelogram rule) and other operations on complex numbers, such as reflections with respect to the real axis (complex conjugate).

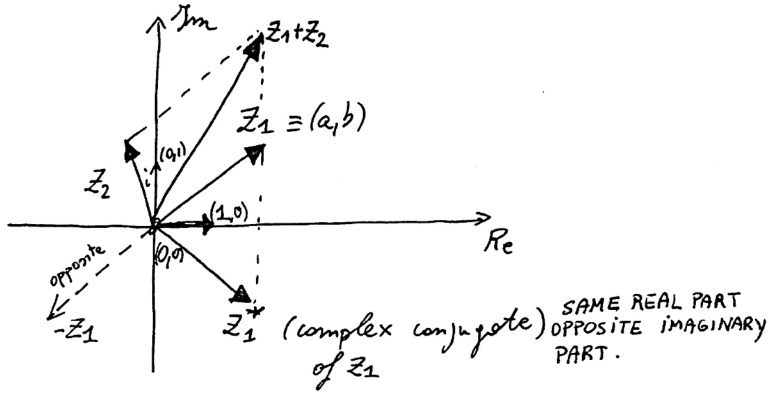
Complex conjugate and modulus of a complex number Let $z = x+iy$ be a complex number. The complex conjugate of z is the complex number

$$z^* = x - iy \qquad \text{(complex conjugate).} \tag{18}$$

Note that z^* has the same real part of z, but opposite imaginary part. Denote the real and imaginary parts of $z = x + iy$ as

$$\mathrm{Re}(z) = x \quad \text{(real part of } z\text{)} \qquad \mathrm{Im}(z) = y \quad \text{(imaginary part of } z\text{).} \tag{19}$$

With this notation we have that the complex conjugate satisfies the following properties:



Theorem 1 (Properties of the complex conjugate)**.** Let $z, w \in \mathbb{C}$ be two arbitrary complex numbers. Then

1. $(z^*)^* = z$
2. $(z+w)^* = z^* + w^*$
3. $(zw)^* = z^* w^*$
4. $z + z^* = 2\,\mathrm{Re}(z)$
5. $z - z^* = 2i\,\mathrm{Im}(z)$
6. $zz^* = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ (squared modulus of a complex number)
7. $z = z^* \Leftrightarrow z \in \mathbb{R}$

Proof. Let us prove property 3, property 4, and property 6. The proof of the other properties is left as exercise. Let $z = x + iy$ and $w = a + ib$ be two arbitrary complex numbers.

Property 3.

$$(zw)^* = ((x+iy)(a+ib))^* = (xa-yb+i(xb+ya))^* = xa-yb-i(xb+ya) = (x-iy)(a-ib) = z^*w^*.$$

Property 4.

$$z + z^* = (x+iy) + (x+iy)^* = 2x + iy - iy = 2\,\mathrm{Re}(z).$$

Property 6.

$$zz^* = (x+iy)(x+iy)^* = (x+iy)(x-iy) = x^2 + y^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2.$$

□

Remark: By using the complex conjugate, it is easy to obtain the algebraic form of a quotient between two complex numbers. We know that such quotient is a complex number[2], and therefore it can written as $x + iy$. The question is what is x and what is y?. There is a shortcut to answer this

[2] In fact for any $z_1, z_2 \in \mathbb{C}$ we have that z_1/z_2 is the multiplication of z_1 by the inverse of z_2 (which is a complex number). We not that multiplication between two complex numbers is a complex number. Therefore z_1/z_2 is a complex number that can be written in the algebraic form $x + iy$.

question. In practice, given any quotient between two complex numbers z_1 and z_2, we can multiply it the fraction by $z_2^*/z_2^* = 1$ to obtain the algebraic form

$$\frac{z_1}{z_2} = \frac{z_2^*}{z_2^*}\frac{z_1}{z_2} = \frac{z_2^* z_1}{z_2 z_2^*} \tag{20}$$

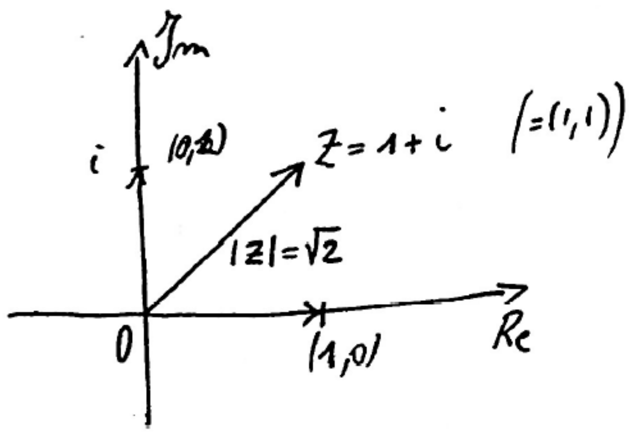
The denominator in (20) a real number (property 6. above). By using this procedure we obtain, for example

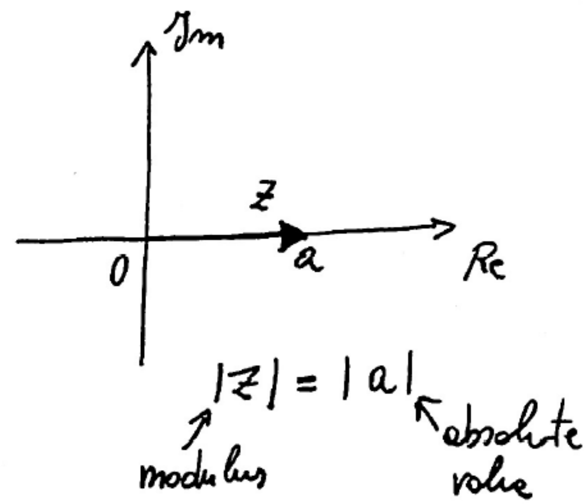
$$\frac{3-2i}{-1+2i} = \frac{(3-2i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{(3-2i)(-1-2i)}{5} = -\frac{7}{5} - \frac{4}{5}i$$

The **modulus** of a complex number $z = x + iy$ is a real number defined as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{zz^*} \qquad \text{(modulus z)}$$

The modulus of z represents the length of the vector defined by the point (x, y) in the complex plane.





Similarly to the complex conjugate, the modulus of a complex number satisfies a certain number of properties which are summarized in the following theorem.

Theorem 2 (Properties of the modulus)**.** Let $z, w \in \mathbb{C}$ be two arbitrary complex numbers. Then,

1. $|z| = 0 \Leftrightarrow z = 0$
2. $|z^*| = |z|$
3. $\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} \leq |z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$
4. $|zw| = |z||w|$
5. $|z + w| \leq |z| + |w|$ (triangle inequality)
6. $||z| - |w|| \leq |z + w|$ (reverse triangle inequality)

Proof. Let us prove property 4, and property 5. The proof of the other properties is left as exercise. Let z and w be two arbitrary complex numbers.

Property 4.
$$|zw|^2 = zwz^*w^* = zz^*ww^* = |z|^2|w|^2 \quad \Rightarrow \quad |zw| = |z||w|.$$

Property 5.

$$\begin{aligned} |z+w|^2 &= (z+w)(z^*+w^*) \\ &= zz^* + ww^* + wz^* + w^*z \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(wz^*). \end{aligned} \tag{21}$$

At this point we notice that[3]

$$\mathrm{Re}(wz^*)^2 = |wz^*|^2 - \mathrm{Im}(wz^*)^2 \leq |wz^*|^2 = |wz|^2 = |w|^2|z|^2. \tag{22}$$

A substitution of this equation into (21) yields

$$|z+w|^2 \leq |z|^2 + |w|^2 + 2|w|^2|z|^2 = (|z| + |w|)^2. \tag{23}$$

By taking the square root of (23) we obtain Property 5.

$\square$

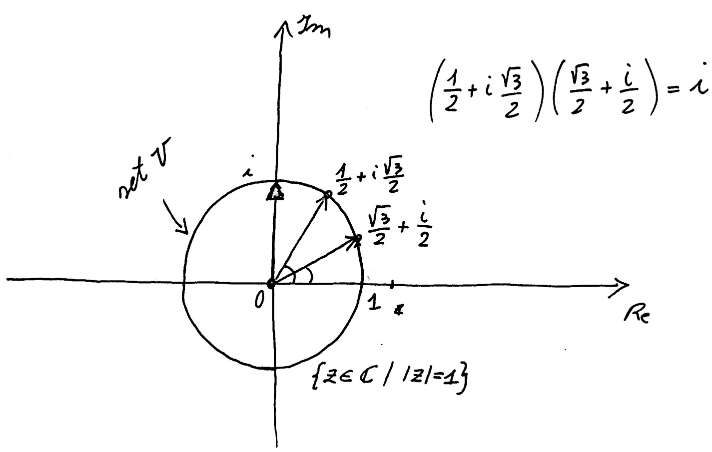
Polar form of a complex number. We have seen in Theorem 2 (property 4) that given two arbitrary complex numbers the norm of their product is equal to the product of their norms, i.e.,

$$|zw| = |z||w| \qquad \forall z, w \in \mathbb{C}. \tag{24}$$

This implies implies that the product of two complex numbers with modulus one is still a complex number with modulus one. In other words, the set

$$\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\} \qquad \text{(unit circle in the complex plane)} \tag{25}$$

is closed under multiplication.



[3]Equation (22) follows from property 6 in Theorem 1, and property 2 and 4 in Theorem 2.

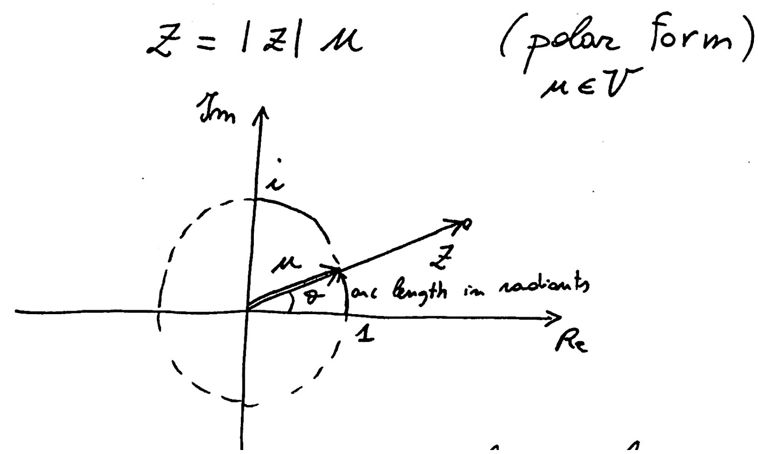
The inverse of a complex number on the unit circle coincides with the conjugate. In fact,

$$|z|^2 = 1 \quad \Rightarrow \quad zz^* = 1 \quad \Rightarrow \quad z^* = \frac{1}{z}. \tag{26}$$

Clearly, by using elements of the set $\mathbb{U}$ defined in (25) we can represent any complex number as

$$z = |z|u(\vartheta) \qquad u(\vartheta) \in \mathbb{U} \tag{27}$$

Note that $u(\vartheta)$ depends only one parameter, i.e., the angle ϑ.



Moreover, by using well known results of trigonometry we can write the complex number $u(\vartheta)$ as

$$u(\vartheta) = \cos(\vartheta) + i \sin(\vartheta). \tag{28}$$

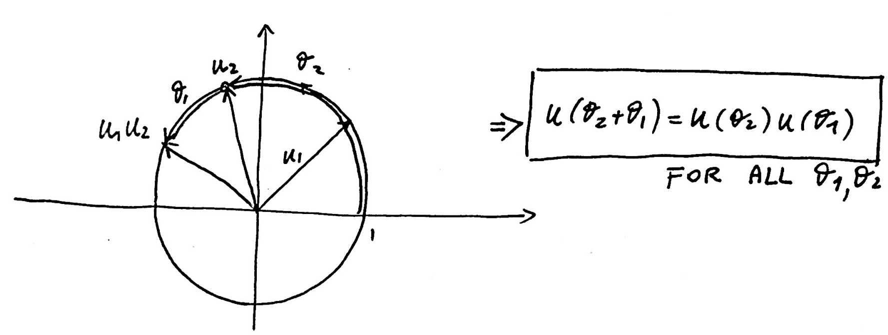
At this point, consider two arbitrary complex numbers on the unit circle

$$u(\vartheta_1) = \cos(\vartheta_1) + i \sin(\vartheta_1) \qquad u(\vartheta_2) = \cos(\vartheta_2) + i \sin(\vartheta_2) \tag{29}$$

and take their product

$$\begin{aligned} u(\vartheta_1)u(\vartheta_2) &= \cos(\vartheta_1)\cos(\vartheta_2) - \sin(\vartheta_1)\sin(\vartheta_2) + i\left[\sin(\vartheta_1)\cos(\vartheta_2) + i\cos(\vartheta_1)\sin(\vartheta_2)\right] \\ &= \cos(\vartheta_1 + \vartheta_2) + i \sin(\vartheta_1 + \vartheta_2) \\ &= u(\vartheta_1 + \vartheta_2) \end{aligned} \tag{30}$$

Remark: This means that the function $u(\vartheta)$ defined in (28) transforms sums into products, i.e., $u(\vartheta_1 + \vartheta_2) = u(\vartheta_1)u(\vartheta_2)$. The similarity between the function $u(\theta)$ and the real exponential function



e^x ($x \in \mathbb{R}$) is remarkable. In fact, we have

$$e^{x_1+x_2} = e^{x_1} e^{x_2} \qquad \forall x_1, x_2 \in \mathbb{R}. \tag{31}$$

This suggests the following definition of *complex exponential function*

$$e^{i\vartheta} = \cos(\vartheta) + i \sin(\vartheta) \tag{32}$$

Remark: There are several other reasons supporting the definition of complex exponential (32). For instance, consider the Taylor series of the real exponential function

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} \tag{33}$$

It is known that such series converges for all $x \in \mathbb{R}$. By substituting x with $i\vartheta$ in (33) we obtain

$$\begin{aligned} e^{i\vartheta} &= \sum_{k=1}^{\infty} \frac{i^k \vartheta^k}{k!} \\ &= \left(1 - \frac{\vartheta^2}{2} + \frac{\vartheta^4}{24} - \cdots\right) + i\left(\vartheta - \frac{\vartheta^3}{6} + \frac{\vartheta^5}{120} - \cdots\right) \\ &= \cos(\vartheta) + i \sin(\vartheta) \end{aligned} \tag{34}$$

In fact, recall that the Taylor series of $\cos(\vartheta)$ and $\sin(\vartheta)$ are

$$\cos(\vartheta) = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \cdots \qquad \sin(\vartheta) = \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} - \cdots . \tag{35}$$

Moreover, in (34) we have

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad \ldots \tag{36}$$

Another reason why it makes sense to define the complex exponential as in (32) is that

$$\frac{de^{i\vartheta}}{d\vartheta} = ie^{i\vartheta}. \tag{37}$$

This can be verified by calculating the derivatives of the right hand side of (32) with respect to ϑ.

In summary, the complex exponential function (32) has the same properties of the real exponential function, e.g., Taylor expansion, derivatives, and the product rule

$$e^{i(\vartheta_1+\vartheta_2)} = e^{i\vartheta_1} e^{i\vartheta_2} \tag{38}$$

By using equation (32) is straightforward to express $\sin(\vartheta)$ and $\cos(\vartheta)$ in terms of complex exponential functions. To this end, we first evaluate (32) at $-\vartheta$

$$e^{-i\vartheta} = \cos(\vartheta) - i \sin(\vartheta). \tag{39}$$

Then we add and subtract (39) to (32) to obtain

$$\cos(\vartheta) = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2} \qquad \sin(\vartheta) = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i} \qquad \text{(Euler's formulas)} \tag{40}$$

Argument of a complex number. We have seen that an arbitrary complex number $z \in \mathbb{C}$ can be written in three equivalent forms:

1. $z = x + iy$ (algebraic form)
2. $z = |z|e^{i\vartheta}$ (polar form)
3. $z = |z|\left(\cos(\vartheta) + i\sin(\vartheta)\right)$ (trigonometric form)

The real number ϑ is called *argument* of the complex number z, and it represents the arclength (in radiants) identified by the point $z/|z|$ on the unit circle $\mathbb{U}$ (see Eq. (25)). To calculate the argument of z, consider the following relations between algebraic form of z and the trigonometric form

$$x = |z|\cos(\vartheta) \qquad y = |z|\sin(\vartheta) \tag{41}$$

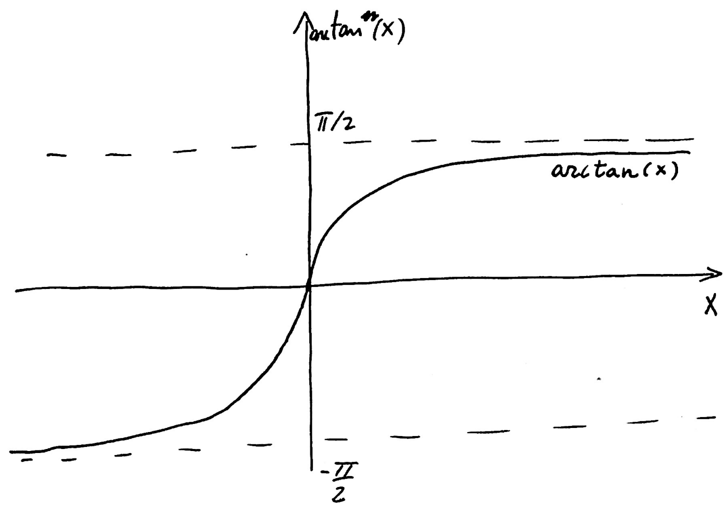
The ratio y/x coincides with the tangent of ϑ

$$\tan(\vartheta) = \frac{y}{x} \tag{42}$$

How do we extract the angle ϑ from the previous equation? One possibility is to use the inverse of the tangent function, i.e., $\arctan(\cdot)$, and write

$$\vartheta = \arctan\left(\frac{y}{x}\right) \tag{43}$$

The problem with this simple approach is that function $\arctan(x)$ is defined only in the open interval $]-\pi/2, \pi/2[$. Hence, the expression (43) can be used only to compute the argument of complex



number with strictly positive real part[4]. To compute the argument of arbitrary complex number $x = x + iy$ we need to shift $\arctan(y/x)$ by π if the real part x is negative

$$\arg(z) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & x > 0 \\ \dfrac{\pi}{2}\operatorname{sign}(y) & x = 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & x < 0 \end{cases} \tag{44}$$

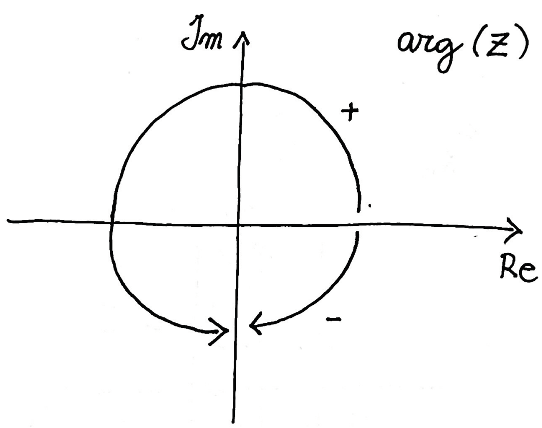
[4] Complex numbers with argument $\vartheta \in]-\pi/2, \pi/2[$ are either in first quadrant ($\vartheta \in [0, \pi/2[$) or in the fourth quadrant ($\vartheta \in]-\pi/2, 0]$).

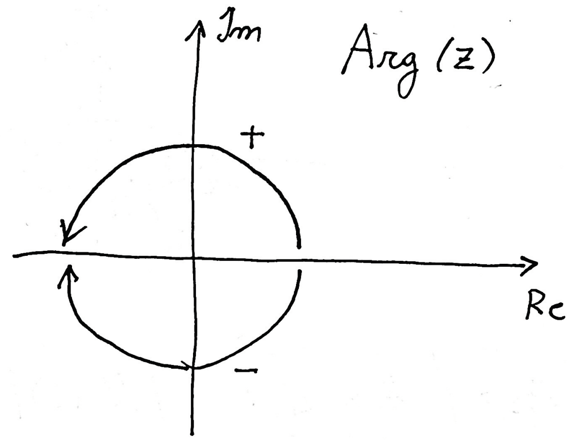
With this definition $\vartheta = \arg(z)$ is unique for all $z \in \mathbb{C}$ and it ranges in $[-\pi/2, 3\pi/2[$.

Alternatively, we can define the argument as (note that here we use capitalized $\mathrm{Arg}(\cdot)$ to distinguish it from (44))

$$\mathrm{Arg}(z) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & x > 0 \\ \dfrac{\pi}{2}\mathrm{sign}(y) & x = 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & x < 0, y \geq 0 \\ \arctan\left(\dfrac{y}{x}\right) - \pi & x < 0, y < 0 \end{cases} \tag{45}$$

With this definition $\vartheta = \mathrm{Arg}(z)$ is unique for all $z \in \mathbb{C}$ and it ranges in $[-\pi, \pi[$.





Remark: If we shift the argument of a complex number by $2k\pi$ ($k \in \mathbb{Z}$, the number is not going to change. In other words,

$$z = 3e^{i\pi/3} \qquad z = 3e^{13i\pi/3} \qquad z = 3e^{-5i\pi/3} \tag{46}$$

are the same complex number. This is due to the 2π-periodicity of the circular functions defining the complex exponential (33).

Integer powers of a complex number (De Moivre's formula). Consider a complex number z expressed in a polar form

$$z = |z|e^{i\vartheta}, \tag{47}$$

where $|z|$ is the modulus of z and ϑ denotes its argument. By multiplying z recursively by itself we obtain

$$z^2 = |z|^2 e^{2i\vartheta}, \qquad z^2 = |z|^3 e^{3i\vartheta}, \quad \ldots. \tag{48}$$

Similarly,

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{|z|}{|z|^2}e^{-i\vartheta} = \frac{1}{|z|}e^{-i\vartheta} = |z|^{-1}e^{-i\vartheta} \tag{49}$$

By multiplying $1/z$ recursively by itself we obtain

$$z^{-2} = |z|^{-2}e^{-2i\vartheta}, \qquad z^{-3} = |z|^{-3}e^{-3i\vartheta}, \quad \ldots. \tag{50}$$

Therefore we proved the following theorem.

Theorem 3 (De Moivre's formula)**.** Let z be any complex number with modulus $|z|$ and argument ϑ. Then

$$z^n = |z|^n e^{in\vartheta} \qquad \forall n \in \mathbb{Z}. \tag{51}$$

Remark: The powers of a complex number complex are points on a spiral in the complex plane. In fact, that the parametric form of a spiral in the Cartesian plane is

$$x(t) = a^t \cos(bt) \qquad y(t) = a^t \sin(bt), \tag{52}$$

where t is the spiral parameter, and a, b are fixed real numbers. These equations coincide with the real and imaginary parts of the powers of z. In fact,

$$\mathrm{Re}(z^n) = |z|^n \cos(n\vartheta) \qquad \mathrm{Im}(z^n) = |z|^n \sin(n\vartheta) \qquad n \in \mathbb{Z}. \tag{53}$$