Lecture 2: Complex numbers

The quadratic equation

\[ z^2 + 1 = 0 \]  \hspace{1cm} (1)

has no solution in \( \mathbb{R} \). In fact, there is no real number such that \( z^2 = -1 \) (recall that the square of any real number is either positive or equal to zero). We can still define solutions to equation (1), but we have to seek them in a different set of numbers. In particular, such a set must include new types of numbers the square of which is negative and real. These numbers are called \textit{imaginary numbers}.

Let "\( i \)" one of such numbers, i.e., an imaginary number defined as \( i^2 = -1 \), or equivalently

\[ i = \sqrt{-1}. \]  \hspace{1cm} (2)

Clearly, \( z = i \) is a solution of equation (1). In fact,

\[ z^2 + 1 = i^2 + 1 = -1 + 1 = 0. \]  \hspace{1cm} (3)

Moreover, \( z = -i \) is another solution of (1) since

\[ z^2 + 1 = (-i)^2 + 1 = (-1)^2i^2 + 1 = -1 + 1 = 0. \]  \hspace{1cm} (4)

Next, consider the polynomial equation

\[ z^2 + z + 1 = 0 \]  \hspace{1cm} (5)

You certainly know that the two roots of this equation can be computed based on the formula

\[ z_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}. \]  \hspace{1cm} (6)

Although the number within the square root is negative (the parabola \( y = z^2 + 2z + 1 \) has no intersection with the axis \( y = 0 \)) we can write (6) as

\[ z_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}. \]  \hspace{1cm} (7)

This suggests that the new set of numbers we are interested in is of the form

\[ z = x + iy \quad \text{(complex number)} \]  \hspace{1cm} (8)

where \( x \) and \( y \) are real numbers and \( i \) is the imaginary unit defined in (2). Specifically, \( x \) is called the \textit{real part} of \( z \), and \( y \) is called the \textit{imaginary part} of \( z \).

Numbers of the form (8) are called \textit{complex numbers}. The set of all complex numbers will be denoted by

\[ \mathbb{C} = \{ z = x + iy : x, y \in \mathbb{R} \}. \]  \hspace{1cm} (9)

As we shall see hereafter, \( \mathbb{C} \) is an algebraic field, i.e., is possible to define in \( \mathbb{C} \) addition and multiplication operations satisfying the same axioms of we have seen in Lecture 1 for \( \mathbb{R} \) (field axioms).
Addition and Multiplication in \( \mathbb{C} \). Consider the following complex numbers

\[
    z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2. \tag{10}
\]

It is natural to define addition and multiplication in \( \mathbb{C} \) by using the addition and multiplication operations we defined in \( \mathbb{R} \) (see Lecture 1). Specifically, we define

\[
    z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \quad \text{(addition operation)} \tag{11}
\]

Note that \( z_1 + z_2 \) is still of the form \( x + iy \) (with \( x = (x_1 + x_2) \) and \( y = (y_1 + y_2) \)), and therefore \( \mathbb{C} \) is closed\(^1\) under the addition operation “+” defined in (11). Similarly,

\[
    z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad \text{(multiplication operation)} \tag{12}
\]

Again, \( z_1 z_2 \) is a complex number, i.e., a number of the form \( x + iy \) (with \( x = (x_1 x_2 - y_1 y_2) \) and \( y = (x_1 y_2 + x_2 y_1) \)). This means that \( \mathbb{C} \) is closed under the multiplication operation defined in (12).

It is easy to show that the set of complex numbers \( \mathbb{C} \), with the addition and multiplication operations defined in (11) and (12) is a field. In other words, for all \( z_1, z_2, z_3 \in \mathbb{C} \) we have that:

1. Addition and multiplication are commutative

\[
    z_1 + z_2 = z_2 + z_1 \quad z_1 z_2 = z_2 z_1 \tag{13}
\]

2. Addition and multiplication in are associative

\[
    (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad (z_1 z_2)z_3 = z_1(z_2 z_3) \tag{14}
\]

3. The distributive property of multiplication relative to addition holds

\[
    z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \tag{15}
\]

4. There exists neutral elements for both addition and multiplication

\[
    z_1 + z_2 = z_1 \quad \Rightarrow \quad z_2 = 0 + i0 \quad \text{(additive neutral element)}
\]

\[
    z_1 z_2 = z_1 \quad \Rightarrow \quad z_2 = 1 + i0 \quad \text{(multiplicative neutral element)} \tag{16}
\]

Based on the definition of additive and multiplicative neutrals it is straightforward to define the opposite and the inverse of a complex number. To this end, let \( z = x + y \)

\[
    z + z_1 = 0 + 0i \quad \Rightarrow \quad z_1 = -x - iy \quad \text{(opposite of } z) \tag{17}
\]

\[
    z z_2 = 1 + 0i \quad \Rightarrow \quad z_2 = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} \quad \text{(inverse of } z) \tag{17}
\]

Let us denote \( z_1 \) as \(-z\) and \( z_2 \) as \(1/z\). Note that equations (17) allow us to define subtraction and division between two complex numbers in terms of addition and multiplication. In fact, subtraction

\(^1\)We say that \( \mathbb{C} \) is closed under the addition operation “+” defined in (11) if for all \( z_1, z_2 \in \mathbb{C} \) we have that \((z_1 + z_2) \in \mathbb{C} \).
of \( z_2 \) from \( z_1 \) is the same as adding the opposite of \( z_2 \) to \( z_1 \). Similarly, \( z_1/z_2 \) is the same as multiplying \( z_1 \) by the inverse of \( z_2 \).

Remark: We have now set up all the machinery to perform any type of algebraic calculation between complex numbers, including addition, subtraction, multiplication and division.

Remark: From what has been said, it is clear that \( \mathbb{C} \) includes \( \mathbb{R} \) as a subset, i.e., \( \mathbb{R} \subset \mathbb{C} \). This can be seen by noting that real numbers are simply complex numbers with zero imaginary part. Moreover, the addition and multiplication operations we defined in \( \mathbb{C} \), i.e., equations (11)-(12), reduce to addition and multiplication between real numbers if we set to zero the imaginary parts. Hence \( \mathbb{R} \subset \mathbb{C} \).

Remark: \( \mathbb{C} \) is not an ordered field. In other words, it does not make sense to write inequalities between complex numbers.

**Graphical representation of complex numbers.** Clearly, there is a one-to-one correspondence between the complex number \( z = x + iy \) and the real numbers \( x, y \in \mathbb{R} \). This means that \( z \) identifies uniquely \( x \) and \( y \), and conversely \( (x, y) \) identifies uniquely the complex number \( z \). This suggests that we could represent \( (x, y) \) as a point (or a vector) in the Cartesian plane.

When the Cartesian plane is used to represent complex numbers, it is usually called *complex plane*. In this setting, the \( x \)-axis is called *real axis* while the \( y \)-axis is called *imaginary axis*.

The complex plane us to easily visualize additions (parallelogram rule) and other operations on complex numbers, such as reflections with respect to the real axis (complex conjugate).

**Complex conjugate and modulus of a complex number** Let \( z = x + iy \) be a complex number. The complex conjugate of \( z \) is the complex number
\[
z^* = x - iy \quad \text{(complex conjugate)}.
\]

Note that \( z^* \) has the same real part of \( z \), but opposite imaginary part. Denote the real and imaginary parts of \( z = x + iy \) as
\[
\text{Re}(z) = x \quad \text{(real part of } z) \quad \text{Im}(z) = y \quad \text{(imaginary part of } z) \quad \text{(19)}
\]

With this notation we have that the complex conjugate satisfies the following properties:
Theorem 1 (Properties of the complex conjugate). Let $z, w \in \mathbb{C}$ be two arbitrary complex numbers. Then

1. $(z^*)^* = z$
2. $(z + w)^* = z^* + w^*$
3. $(zw)^* = z^*w^*$
4. $z + z^* = 2 \text{Re}(z)$
5. $z - z^* = 2i \text{Im}(z)$
6. $zz^* = \text{Re}(z)^2 + \text{Im}(z)^2$ (squared modulus of a complex number)
7. $z = z^* \iff z \in \mathbb{R}$

Proof. Let us prove property 3, property 4, and property 6. The proof of the other properties is left as exercise. Let $z = x + iy$ and $w = a + ib$ be two arbitrary complex numbers.

Property 3.

$$(zw)^* = ((x + iy)(a + ib))^* = (xa - yb + i(xy + ya))^* = xa - yb - i(xy + ya) = (x - iy)(a - ib) = z^*w^*.$$ 

Property 4.

$$z + z^* = (x + iy) + (x + iy)^* = 2x + iy - iy = 2 \text{Re}(z).$$

Property 6.

$$zz^* = (x + iy)(x + iy)^* = (x + iy)(x - iy) = x^2 + y^2 = \text{Re}(z)^2 + \text{Im}(z)^2.$$
question. In practice, given any quotient between two complex numbers $z_1$ and $z_2$, we can multiply it the fraction by $z_2^*/z_2^* = 1$ to obtain the algebraic form

$$\frac{z_1}{z_2} = \frac{z_2^* z_1}{z_2^* z_2} = \frac{z_2 z_1}{z_2 z_2^*}$$

(20)

The denominator in (20) a real number (property 6. above). By using this procedure we obtain, for example

$$\frac{3 - 2i}{-1 + 2i} = \frac{(3 - 2i)(-1 - 2i)}{(-1 + 2i)(-1 - 2i)} = \frac{3 - 2i}{5} = -\frac{7}{5} - \frac{4}{5}i$$

The modulus of a complex number $z = x + iy$ is a real number defined as

$$|z| = \sqrt{x^2 + y^2} = \sqrt{zz^*}$$

(modulus $z$)

The modulus of $z$ represents the length of the vector defined by the point $(x, y)$ in the complex plane.

Similarly to the complex conjugate, the modulus of a complex number satisfies a certain number of properties which are summarized in the following theorem.

**Theorem 2** (Properties of the modulus). Let $z, w \in \mathbb{C}$ be two arbitrary complex numbers. Then,

1. $|z| = 0 \iff z = 0$
2. $|z^*| = |z|
3. $\{|\text{Re}(z)|, |\text{Im}(z)|\} \leq |z| \leq |\text{Re}(z)| + |\text{Im}(z)|$
4. $|zw| = |z||w|$
5. $|z + w| \leq |z| + |w|$ (triangle inequality)
6. $||z| - |w|| \leq |z + w|$ (reverse triangle inequality)
Proof. Let us prove property 4, and property 5. The proof of the other properties is left as exercise. Let $z$ and $w$ be two arbitrary complex numbers.

Property 4.

$$|zw|^2 = zw^*w^* = zz^*ww^* = |z|^2|w|^2 \Rightarrow |zw| = |z||w|.$$  

Property 5.

$$|z + w|^2 = (z + w)(z^* + w^*)$$
$$= zz^* + ww^* + wz^* + wz$$
$$= |z|^2 + |w|^2 + 2 \text{Re}(wz^*).$$  \hspace{1cm} (21)

At this point we notice that$^3$

$$\text{Re}(wz^*)^2 = |wz^*|^2 - \text{Im}(wz^*)^2 \leq |wz|^2 = |w|^2|z|^2.$$  \hspace{1cm} (22)

A substitution of this equation into (21) yields

$$|z + w|^2 \leq |z|^2 + |w|^2 + 2|w|^2|z|^2 = (|z| + |w|)^2.$$  \hspace{1cm} (23)

By taking the square root of (23) we obtain Property 5.

\[\square\]

**Polar form of a complex number.** We have seen in Theorem 2 (property 4) that given two arbitrary complex numbers the norm of their product is equal to the product of their norms, i.e.,

$$|zw| = |z||w| \quad \forall z, w \in \mathbb{C}.$$  \hspace{1cm} (24)

This implies implies that the product of two complex numbers with modulus one is still a complex number with modulus one. In other words, the set

$$U = \{z \in \mathbb{C} : |z| = 1\} \quad \text{(unit circle in the complex plane)}$$  \hspace{1cm} (25)

is closed under multiplication.

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$^3$Equation (22) follows from property 6 in Theorem 1, and property 2 and 4 in Theorem 2.
The inverse of a complex number on the unit circle coincides with the conjugate. In fact,

$$|z|^2 = 1 \Rightarrow zz^* = 1 \Rightarrow z^* = \frac{1}{z}. \quad (26)$$

Clearly, by using elements of the set $\mathbb{U}$ defined in (25) we can represent any complex number as

$$z = |z|u(\vartheta) \quad u(\vartheta) \in \mathbb{U} \quad (27)$$

Note that $u(\vartheta)$ depends only one parameter, i.e., the angle $\vartheta$.

Moreover, by using well known results of trigonometry we can write the complex number $u(\vartheta)$ as

$$u(\vartheta) = \cos(\vartheta) + i\sin(\vartheta). \quad (28)$$

At this point, consider two arbitrary complex numbers on the unit circle

$$u(\vartheta_1) = \cos(\vartheta_1) + i\sin(\vartheta_1) \quad u(\vartheta_2) = \cos(\vartheta_2) + i\sin(\vartheta_2) \quad (29)$$

and take their product

$$u(\vartheta_1)u(\vartheta_2) = \cos(\vartheta_1)\cos(\vartheta_2) - \sin(\vartheta_1)\sin(\vartheta_2) + i[\sin(\vartheta_1)\cos(\vartheta_2) + i\cos(\vartheta_1)\sin(\vartheta_2)]$$

$$= \cos(\vartheta_1 + \vartheta_2) + i\sin(\vartheta_1 + \vartheta_2)$$

$$= u(\vartheta_1 + \vartheta_2) \quad (30)$$

Remark: This means that the function $u(\vartheta)$ defined in (28) transforms sums into products, i.e., $u(\vartheta_1 + \vartheta_2) = u(\vartheta_1)u(\vartheta_2)$. The similarity between the function $u(\vartheta)$ and the real exponential function
$e^x$ ($x \in \mathbb{R}$) is remarkable. In fact, we have
\[ e^{x_1+x_2} = e^{x_1}e^{x_2} \quad \forall x_1, x_2 \in \mathbb{R}. \] (31)

This suggests the following definition of complex exponential function
\[ e^{i\vartheta} = \cos(\vartheta) + i\sin(\vartheta) \] (32)

**Remark:** There are several other reasons supporting the definition of complex exponential (32). For instance, consider the Taylor series of the real exponential function
\[ e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} \] (33)

It is known that such series converges for all $x \in \mathbb{R}$. By substituting $x$ with $i\vartheta$ in (33) we obtain
\[ e^{i\vartheta} = \sum_{k=1}^{\infty} \frac{i^k\vartheta^k}{k!} = \left(1 - \frac{\vartheta^2}{2} + \frac{\vartheta^4}{24} - \cdots \right) + i \left(\vartheta - \frac{\vartheta^3}{6} + \frac{\vartheta^5}{120} - \cdots \right) = \cos(\vartheta) + i\sin(\vartheta) \] (34)

In fact, recall that the Taylor series of $\cos(\vartheta)$ and $\sin(\vartheta)$ are
\[ \cos(\vartheta) = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \cdots \quad \sin(\vartheta) = \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} - \cdots. \] (35)

Moreover, in (34) we have
\[ i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad \ldots \] (36)

Another reason why it makes sense to define the complex exponential as in (32) is that
\[ \frac{d e^{i\vartheta}}{d\vartheta} = i e^{i\vartheta}. \] (37)

This can be verified by calculating the derivatives of the right hand side of (32) with respect to $\vartheta$.

In summary, the complex exponential function (32) has the same properties of the real exponential function, e.g., Taylor expansion, derivatives, and the product rule
\[ e^{i(\vartheta_1+\vartheta_2)} = e^{i\vartheta_1}e^{i\vartheta_2} \] (38)

By using equation (32) is straightforward to express $\sin(\vartheta)$ and $\cos(\vartheta)$ in terms of complex exponential functions. To this end, we first evaluate (32) at $-\vartheta$
\[ e^{-i\vartheta} = \cos(\vartheta) - i\sin(\vartheta). \] (39)

Then we add and subtract (39) to (32) to obtain
\[ \cos(\vartheta) = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2} \quad \sin(\vartheta) = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i} \] (Euler’s formulas) (40)
**Argument of a complex number.** We have seen that an arbitrary complex number $z \in \mathbb{C}$ can be written in three equivalent forms:

1. $z = x + iy$ (algebraic form)
2. $z = |z|e^{i\vartheta}$ (polar form)
3. $z = |z| (\cos(\vartheta) + i \sin(\vartheta))$ (trigonometric form)

The real number $\vartheta$ is called argument of the complex number $z$, and it represents the arclength (in radians) identified by the point $z/|z|$ on the unit circle $U$ (see Eq. (25)). To calculate the argument of $z$, consider the following relations between algebraic form of $z$ and the trigonometric form

\[ x = |z| \cos(\vartheta) \quad y = |z| \sin(\vartheta) \quad (41) \]

The ratio $y/x$ coincides with the tangent of $\vartheta$

\[ \tan(\vartheta) = \frac{y}{x} \quad (42) \]

How do we extract the angle $\vartheta$ from the previous equation? One possibility is to use the inverse of the tangent function, i.e., $\arctan(\cdot)$, and write

\[ \vartheta = \arctan \left( \frac{y}{x} \right) \quad (43) \]

The problem with this simple approach is that function $\arctan(x)$ is defined only in the open interval $]-\pi/2, \pi/2[$. Hence, the expression (43) can be used only to compute the argument of complex number with strictly positive real part\(^4\). To compute the argument of arbitrary complex number $x = x + iy$ we need to shift $\arctan(y/x)$ by $\pi$ if the real part $x$ is negative

\[ \text{arg}(z) = \begin{cases} \arctan \left( \frac{y}{x} \right) & x > 0 \\ \frac{\pi}{2} \text{sign}(y) & x = 0 \\ \arctan \left( \frac{y}{x} \right) + \pi & x < 0 \end{cases} \quad (44) \]

\(^4\)Complex numbers with argument $\vartheta \in ]-\pi/2, \pi/2[$ are either in first quadrant ($\vartheta \in [0, \pi/2]$) or in the fourth quadrant ($\vartheta \in ]-\pi/2, 0]$).
With this definition $\vartheta = \arg(z)$ is unique for all $z \in \mathbb{C}$ and it ranges in $[-\pi/2, 3\pi/2]$.

Alternatively, we can define the argument as (note that here we use capitalized $\text{Arg}(\cdot)$ to distinguish it from (44))

$$
\text{Arg}(z) = \begin{cases} 
\arctan\left(\frac{y}{x}\right) & x > 0 \\
\frac{\pi}{2}\text{sign}(y) & x = 0 \\
\arctan\left(\frac{y}{x}\right) + \pi & x < 0, y \geq 0 \\
\arctan\left(\frac{y}{x}\right) - \pi & x < 0, y < 0
\end{cases}
$$

With this definition $\vartheta = \text{Arg}(z)$ is unique for all $z \in \mathbb{C}$ and it ranges in $[-\pi, \pi]$.

**Remark:** If we shift the argument of a complex number by $2k\pi$ ($k \in \mathbb{Z}$, the number is not going to change. In other words,

$$
z = 3e^{i\pi/3} \quad z = 3e^{13i\pi/3} \quad z = 3e^{-5i\pi/3}
$$

are the same complex number. This is due to the $2\pi$-periodicity of the circular functions defining the complex exponential (33).

**Integer powers of a complex number (De Moivre’s formula).** Consider a complex number $z$ expressed in a polar form

$$
z = |z|e^{i\vartheta},
$$

where $|z|$ is the modulus of $z$ and $\vartheta$ denotes its argument. By multiplying $z$ recursively by itself we obtain

$$
z^2 = |z|^2e^{2i\vartheta}, \quad z^3 = |z|^3e^{3i\vartheta}, \quad \ldots.
$$

Similarly,

$$
z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{|z|^2}{|z|^2}e^{-i\vartheta} = \frac{1}{|z|}e^{-i\vartheta} = |z|^{-1}e^{-i\vartheta}
$$

By multiplying $1/z$ recursively by itself we obtain

$$
z^{-2} = |z|^{-2}e^{-2i\vartheta}, \quad z^{-3} = |z|^{-3}e^{-3i\vartheta}, \quad \ldots.
$$

Therefore we proved the following theorem.
Theorem 3 (De Moivre’s formula). Let $z$ be any complex number with modulus $|z|$ and argument $\vartheta$. Then

$$z^n = |z|^n e^{in\vartheta} \quad \forall n \in \mathbb{Z}.$$  \hfill (51)

Remark: The powers of a complex number complex are points on a spiral in the complex plane. In fact, that the parametric form of a spiral in the Cartesian plane is

$$x(t) = at \cos(bt) \quad y(t) = at \sin(bt),$$  \hfill (52)

where $t$ is the spiral parameter, and $a, b$ are fixed real numbers. These equations coincide with the real and imaginary parts of the powers of $z$. In fact,

$$\text{Re}(z^n) = |z|^n \cos(n\vartheta) \quad \text{Im}(z^n) = |z|^n \sin(n\vartheta) \quad n \in \mathbb{Z}.$$  \hfill (53)