Lecture 2: Complex numbers

The quadratic equation

$$z^2 + 1 = 0 \tag{1}$$

has no solution in $\mathbb{R}$. In fact, there is no real number such that $z^2 = -1$ (recall that the square of any real number is either positive or equal to zero). However, we can still define solutions of equation (1), but we have to seek them in a different set of numbers. In particular, such a set must include new types of numbers the square of which is a negative real number. These numbers are called *imaginary numbers*.

Let "i" be one of such numbers, i.e., an imaginary number defined as

$$i = \sqrt{-1}. \tag{2}$$

Clearly, $z = i$ is a solution of equation (1). In fact,

$$z^2 + 1 = i^2 + 1 = -1 + 1 = 0. \tag{3}$$

Moreover, $z = -i$ is another solution of equation (1) since

$$z^2 + 1 = (-i)^2 + 1 = (-1)^2 i^2 + 1 = -1 + 1 = 0. \tag{4}$$

Next, consider the polynomial equation

$$z^2 + z + 1 = 0. \tag{5}$$

We can rearrange such polynomial equation as

$$\left(z + \frac{1}{2}\right)^2 + \frac{3}{4} = 0. \tag{6}$$

Upon definition of

$$u = z + \frac{1}{2} \tag{7}$$

we can write (6) as

$$u^2 = -\frac{3}{4} \quad \Leftrightarrow \quad u_{1,2} = \pm i\frac{\sqrt{3}}{2}. \tag{8}$$

Substituting $u_{1,2}$ back into (7) yields the following two solutions to equation (5)

$$z_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \tag{9}$$

This suggests that the new set of numbers we are interested in (at lest from the viewpoint of solving quadratic polynomial equations) has the form

$$z = x + iy \qquad \text{(complex number)}, \tag{10}$$

where x and y are real numbers and i is the imaginary unit defined in equation (2). Specifically, x is called the *real part* of z, and y is called the *imaginary part* of z. The real and imaginary parts of z are often denotes as

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad \mathrm{Im}(z) = y. \tag{11}$$

Numbers of the form (10) are called *complex numbers*. The set of all complex numbers will be denoted by

$$\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}. \tag{12}$$

As we shall see hereafter, $\mathbb{C}$ is an algebraic field, i.e., is possible to define in $\mathbb{C}$ addition and multiplication operations satisfying the same field axioms we have seen in Lecture 1 for $\mathbb{R}$ (field axioms for real numbers).

Addition and multiplication. Consider the following complex numbers

$$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2. \tag{13}$$

It is natural to define addition and multiplication in $\mathbb{C}$ by using the addition and multiplication operations we defined in $\mathbb{R}$ (see Lecture 1). Specifically, we define

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \qquad \text{(addition operation)} \tag{14}$$

Note that $z_1 + z_2$ is still of the form $x + iy$ (with $x = (x_1 + x_2)$ and $y = (y_1 + y_2)$). Therefore $\mathbb{C}$ is closed[1] under the addition operation "+" defined in (14). Similarly,

$$\begin{aligned} z_1 z_2 =&(x_1 + iy_1)(x_2 + iy_2) \\ =&(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \qquad \text{(multiplication operation)} \end{aligned} \tag{15}$$

Again, z_1z_2 is a complex number, i.e., a number of the form $x + iy$ (with $x = (x_1x_2 - y_1y_2)$ and $y = (x_1y_2 + x_2y_1)$). This means that $\mathbb{C}$ is closed under the multiplication operation defined in (15)

It is easy to show that the set of complex numbers $\mathbb{C}$, with the addition and multiplication operations defined in (14) and (15) is a field. In other words, for all $z_1, z_2, z_3 \in \mathbb{C}$ we have that:

1. Addition and multiplication are *commutative*

$$z_1 + z_2 = z_2 + z_1 \qquad z_1z_2 = z_2z_1 \tag{16}$$

2. Addition and multiplication in are *associative*

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \qquad (z_1z_2)z_3 = z_1(z_2z_3) \tag{17}$$

3. The *distributive property* of multiplication relative to addition holds

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \tag{18}$$

4. There exists *neutral elements* for both addition and multiplication

$$\begin{aligned} &z_1 + z_2 = z_1 \quad \Rightarrow \quad z_2 = 0 + i0 \qquad \text{(additive neutral element)} \\ &z_1z_2 = z_1 \qquad \Rightarrow \quad z_2 = 1 + i0 \qquad \text{(multiplicative neutral element)} \end{aligned} \tag{19}$$

[1]We say that $\mathbb{C}$ is closed under the addition operation $+$ defined in (14) if for all $z_1, z_2 \in \mathbb{C}$ we have that $(z_1 + z_2) \in \mathbb{C}$.

Based on the definition of additive and multiplicative neutrals it is straightforward to define the *opposite* and the *inverse* of a complex number. To this end, let $z = x + y$

$$
\begin{aligned}
z + z_1 = 0 + 0i \quad &\Rightarrow \quad z_1 = -x - iy \qquad \text{(opposite of } z\text{)} \\
zz_2 = 1 + 0i \quad &\Rightarrow \quad z_2 = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \qquad \text{(inverse of } z\text{)}
\end{aligned} \tag{20}
$$

Let us denote z_1 as $-z$ and z_2 as $1/z$. Note that equations (20) allow us to define *subtraction* and *division* between two complex numbers in terms of addition and multiplication. In fact, subtraction of z_2 from z_1 is the same as adding the opposite of z_2 to z_1. Similarly, z_1/z_2 is the same as multiplying z_1 by the inverse of z_2.

Remark: We have now set up all the machinery to perform any type of algebraic calculation between complex numbers, including addition, subtraction, multiplication and division.

Remark: From what has been said, it is clear that $\mathbb{C}$ includes $\mathbb{R}$ as a subset, i.e., $\mathbb{R} \subset \mathbb{C}$. This can be seen by noting that real numbers are simply complex numbers with zero imaginary part. Moreover, the addition and multiplication operations we defined in $\mathbb{C}$, i.e., equations (14)-(15), reduce to addition and multiplication between real numbers if we set to zero the imaginary parts. Hence $\mathbb{R} \subset \mathbb{C}$.

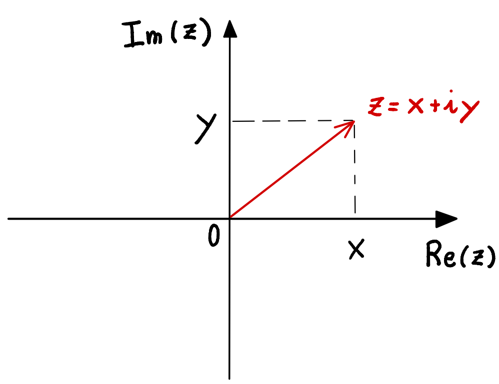
Remark: $\mathbb{C}$ is not an ordered field. In other words, it does not make sense to write inequalities between complex numbers.

Graphical representation of complex numbers. There is a one-to-one correspondence between the complex number

$$z = x + iy$$

and the pair of real numbers $x, y \in \mathbb{R}$. This means that z identifies uniquely x and y, and conversely the pair (x, y) identifies uniquely the complex number z. This suggests that we could represent (x, y) as a point (or a vector) in the Cartesian plane.

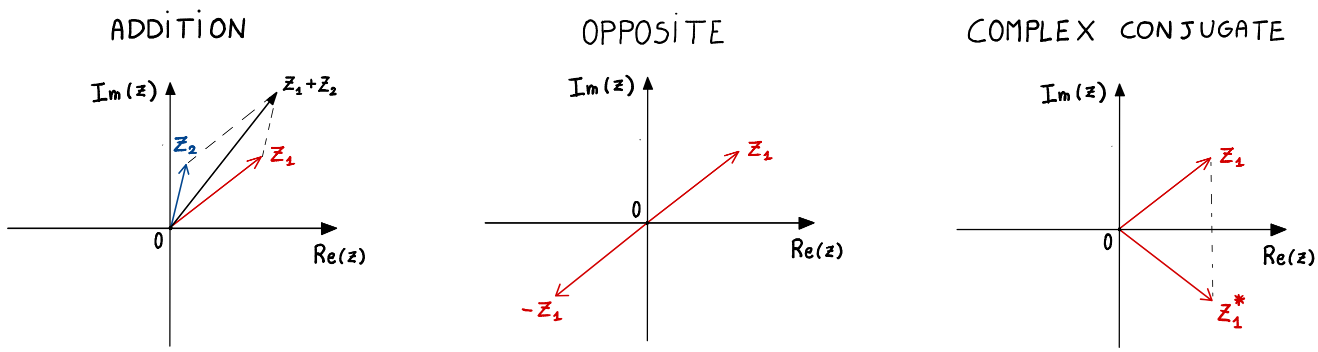
When the Cartesian plane is used to represent complex numbers, it is usually called *complex plane*. In this setting, the x-axis is called *real axis* while the y-axis is called *imaginary axis*.



Recall that for the complex number $z = x + iy$ we defined

$$\mathrm{Re}(z) = x \quad \text{(real part of } z\text{)} \qquad \mathrm{Im}(z) = y \quad \text{(imaginary part of } z\text{)}. \tag{21}$$

The complex plane allows us to easily visualize addition between complex numbers (parallelogram rule), and other operations such as the opposite of a complex number, and the complex conjugate (reflections with respect to the real axis).



Complex conjugate. Let $z = x + iy$ be a complex number. The complex conjugate of z is the complex number

$$z^* = x - iy \qquad \text{(complex conjugate).} \tag{22}$$

Note that z^* has the same real part of z, but opposite imaginary part. With this notation, we have the following characterization of the complex conjugate.

Theorem 1 (Properties of the complex conjugate)**.** Let $z, w \in \mathbb{C}$ be two arbitrary complex numbers. Then

1. $(z^*)^* = z$
2. $(z + w)^* = z^* + w^*$
3. $(zw)^* = z^* w^*$
4. $z + z^* = 2\,\text{Re}(z)$
5. $z - z^* = 2i\,\text{Im}(z)$
6. $zz^* = \text{Re}(z)^2 + \text{Im}(z)^2$
7. $z = z^* \Leftrightarrow z \in \mathbb{R}$

Proof. Let us prove property 3, property 4, and property 6. The proof of the other properties is left as exercise. Let $z = x + iy$ and $w = a + ib$ be two arbitrary complex numbers.

Property 3.

$$(zw)^* = ((x+iy)(a+ib))^* = (xa-yb+i(xb+ya))^* = xa-yb-i(xb+ya) = (x-iy)(a-ib) = z^*w^*.$$

Property 4.

$$z + z^* = (x + iy) + (x + iy)^* = 2x + iy - iy = 2\,\text{Re}(z).$$

Property 6.

$$zz^* = (x + iy)(x + iy)^* = (x + iy)(x - iy) = x^2 + y^2 = \text{Re}(z)^2 + \text{Im}(z)^2.$$

□

Remark: By using the complex conjugate, it is easy to express the quotient between two complex numbers, e.g.,

$$\frac{3-2i}{-1+2i} \tag{23}$$

in an standard algebraic form. We know that such ratio is a complex number[2], and therefore it can written is the form $x+iy$. The question is what is x and what is y? There is a shortcut to answer this question. In practice, given a quotient between two complex numbers z_1 and z_2 (i.e., z_1/z_2) we can multiply the numerator and the denominator by z_2^* to obtain the algebraic form

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*}. \tag{24}$$

The denominator in (24) a real number (by property 6. in Theorem 1).

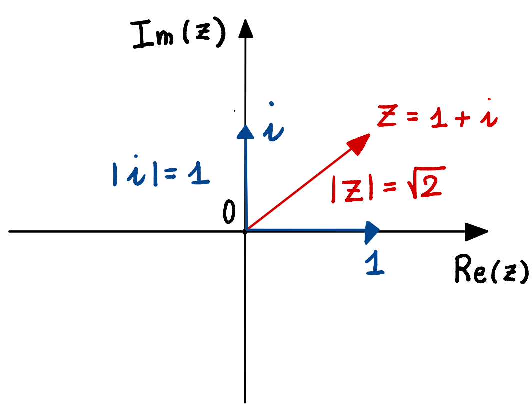
Example: Let $z_1 = 3-2i$ and $z_2 = -1+2i$. Compute the algebraic form of the complex number z_1/z_2. We have

$$\frac{3-2i}{-1+2i} = \frac{(3-2i)(-1-2i)}{\underbrace{(-1+2i)}_{z_1}\underbrace{(-1-2i)}_{z_1^*}} = \frac{(3-2i)(-1-2i)}{5} = -\frac{7}{5} - \frac{4}{5}i.$$

Modulus of a complex number. The modulus of a complex number $z = x+iy$ is a real number defined as

$$|z| = \sqrt{x^2+y^2} = \sqrt{zz^*} \qquad \text{(modulus of } z\text{)}$$

The modulus of z represents the *length* of the vector defined by the point (x,y) in the complex plane.



[2]In fact for any $z_1, z_2 \in \mathbb{C}$ we have that z_1/z_2 is the multiplication of z_1 by the inverse of z_2 (which is a complex number). Recall that multiplication between two complex numbers is a complex number. Therefore z_1/z_2 is a complex number that can be written in the algebraic form $x+iy$.

Clearly, the modulus of the imaginary number i is

$$|i| = |0 + 1i| = \sqrt{0^2 + 1^2} = 1. \tag{25}$$

Similarly, the modulus of the complex number $z = 1 + i$ is

$$|z| = |1 + 1i| = \sqrt{1^2 + 1^2} = \sqrt{2}. \tag{26}$$

The modulus of a complex number satisfies a certain number of properties which are summarized in the following Theorem.

Theorem 2 (Properties of the modulus)**.** Let $z, w \in \mathbb{C}$ be two arbitrary complex numbers. Then,

1. $|z| = 0 \Leftrightarrow z = 0$
2. $|z^*| = |z|$
3. $\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\} \leq |z| \leq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$
4. $|zw| = |z||w|$
5. $|z + w| \leq |z| + |w|$ (triangle inequality)
6. $||z| - |w|| \leq |z + w|$ (reverse triangle inequality)

Proof. Let us prove property 4, and property 5. The proof of the other properties is left as exercise. Let z and w be two arbitrary complex numbers.

Property 4.

$$|zw|^2 = zwz^*w^* = zz^*ww^* = |z|^2|w|^2 \quad \Rightarrow \quad |zw| = |z||w|.$$

Property 5.

$$\begin{aligned} |z + w|^2 &= (z + w)(z^* + w^*) \\ &= zz^* + ww^* + wz^* + w^*z \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(wz^*) \\ &\leq |z|^2 + |w|^2 + 2|\operatorname{Re}(wz^*)|. \end{aligned} \tag{27}$$

At this point we notice that[3]

$$\operatorname{Re}(wz^*)^2 = |wz^*|^2 - \operatorname{Im}(wz^*)^2 \leq |wz^*|^2 = |w||z^*|^2 = |w|^2|z|^2, \tag{28}$$

i.e.,

$$|\operatorname{Re}(wz^*)| \leq |w||z|. \tag{29}$$

A substitution of this equation into (27) yields

$$|z + w|^2 \leq |z|^2 + |w|^2 + 2|w|^2|z|^2 = (|z| + |w|)^2. \tag{30}$$

By taking the square root of (30) we obtain Property 5.

[3] Equation (28) follows from property 6 in Theorem 1, and property 2 and 4 in Theorem 2.

□

Polar form of a complex number. We have seen in Theorem 2 (property 4) that given two arbitrary complex numbers the norm of their product is equal to the product of their norms, i.e.,

$$|zw| = |z||w| \qquad \forall z, w \in \mathbb{C}. \tag{31}$$

This implies implies that the product of two complex numbers with modulus one is still a complex number with modulus one. In other words, the set

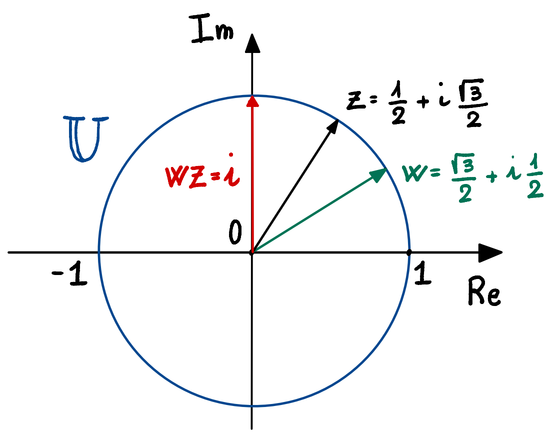
$$\mathbb{U} = \{z \in \mathbb{C} : \ |z| = 1\} \qquad \text{(unit circle in the complex plane)} \tag{32}$$

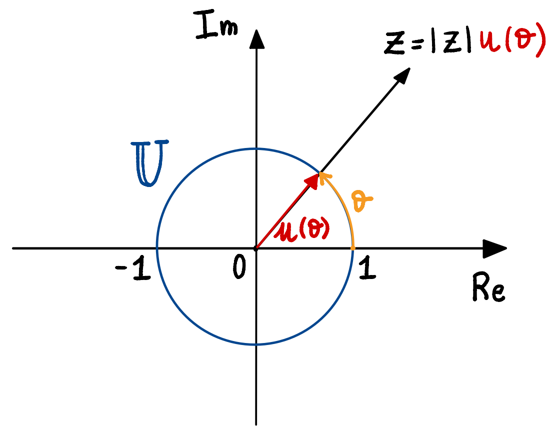
is closed under multiplication. For example, consider the following two complex numbers z and w

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad w = \frac{\sqrt{3}}{2} + i\frac{1}{2} \tag{33}$$

both of which have modulus equal to one (verify it!). Clearly, we have

$$|zw| = |i| = 1. \tag{34}$$





The inverse of a complex number on the unit circle (32) coincides with the complex conjugate. In fact,

$$|z|^2 = 1 \quad \Rightarrow \quad zz^* = 1 \quad \Rightarrow \quad z^* = \frac{1}{z}. \tag{35}$$

Clearly, by using elements of the set $\mathbb{U}$ defined in (32) we can represent any complex number as

$$z = |z|u(\vartheta) \qquad u(\vartheta) \in \mathbb{U}. \tag{36}$$

Note that $u(\vartheta)$ depends only one parameter, i.e., the angle ϑ (arclength on the unit circle). Moreover, by using well-known results of trigonometry we can write the complex number $u(\vartheta)$ as

$$u(\vartheta) = \cos(\vartheta) + i\sin(\vartheta). \tag{37}$$

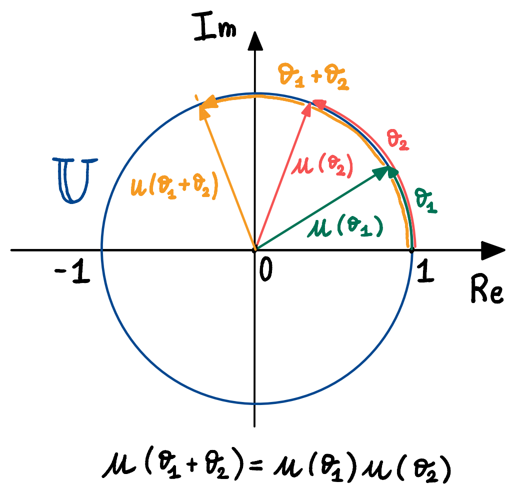
Complex exponential function. Consider two arbitrary complex numbers on the unit circle (32)

$$u(\vartheta_1) = \cos(\vartheta_1) + i\sin(\vartheta_1) \qquad u(\vartheta_2) = \cos(\vartheta_2) + i\sin(\vartheta_2) \tag{38}$$

and take their product

$$\begin{aligned} u(\vartheta_1)u(\vartheta_2) =& \cos(\vartheta_1)\cos(\vartheta_2) - \sin(\vartheta_1)\sin(\vartheta_2) + i\left[\sin(\vartheta_1)\cos(\vartheta_2) + \cos(\vartheta_1)\sin(\vartheta_2)\right] \\ =& \cos(\vartheta_1 + \vartheta_2) + i\sin(\vartheta_1 + \vartheta_2) \\ =& u(\vartheta_1 + \vartheta_2) \end{aligned} \tag{39}$$

Remark: This means that the function $u(\vartheta)$ defined in (37) transforms sums into products, i.e., $u(\vartheta_1 + \vartheta_2) = u(\vartheta_1)u(\vartheta_2)$.



The similarity between the function $u(\vartheta)$ and the real exponential function e^x ($x \in \mathbb{R}$) is quite remarkable. In fact, we have

$$e^{x_1+x_2} = e^{x_1}e^{x_2}, \qquad \text{for all} \quad x_1, x_2 \in \mathbb{R}. \tag{40}$$

This suggests the following definition of *complex exponential function*

$$e^{i\vartheta} = \cos(\vartheta) + i\sin(\vartheta). \tag{41}$$

Remark: There are several other reasons supporting the definition of complex exponential function (41). For instance, consider the Taylor series of the real exponential function

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!}. \tag{42}$$

It is known that such series converges for all $x \in \mathbb{R}$. By substituting x with $i\vartheta$ in (42) we obtain

$$\begin{aligned} e^{i\vartheta} =& \sum_{k=1}^{\infty} \frac{i^k \vartheta^k}{k!} \\ =& \left(1 - \frac{\vartheta^2}{2} + \frac{\vartheta^4}{24} - \cdots\right) + i\left(\vartheta - \frac{\vartheta^3}{6} + \frac{\vartheta^5}{120} - \cdots\right) \\ =& \cos(\vartheta) + i\sin(\vartheta). \end{aligned} \tag{43}$$

In fact, recall that the Taylor series of $\cos(\vartheta)$ and $\sin(\vartheta)$ are

$$\cos(\vartheta) = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \cdots \qquad \sin(\vartheta) = \vartheta - \frac{\vartheta^3}{3!} + \frac{\vartheta^5}{5!} - \cdots . \tag{44}$$

Another reason why it makes sense to define the complex exponential as in (41) is that

$$\frac{de^{i\vartheta}}{d\vartheta} = ie^{i\vartheta}. \tag{45}$$

This can be verified by calculating the derivatives of the right hand side of (41) with respect to ϑ.

In summary, the complex exponential function (41) has the same properties of the real exponential function, e.g., Taylor expansion, derivatives, and the product rule

$$e^{i(\vartheta_1+\vartheta_2)} = e^{i\vartheta_1}e^{i\vartheta_2} \tag{46}$$

Euler's formulas. By using equation (41) is straightforward to express $\sin(\vartheta)$ and $\cos(\vartheta)$ in terms of complex exponential functions. To this end, we first evaluate (41) at $-\vartheta$

$$e^{-i\vartheta} = \cos(\vartheta) - i\sin(\vartheta). \tag{47}$$

Then we add and subtract (47) to (41) to obtain

$$\cos(\vartheta) = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2} \qquad \text{and} \qquad \sin(\vartheta) = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}. \tag{48}$$

Argument of a complex number. We have seen that an arbitrary complex number $z \in \mathbb{C}$ can be written in three equivalent forms:

1. $z = x + iy$ (algebraic form)
2. $z = |z|e^{i\vartheta}$ (polar form)
3. $z = |z|\left(\cos(\vartheta) + i\sin(\vartheta)\right)$ (trigonometric form)

The real number ϑ is called *argument* of the complex number z, and it represents the arclength (in radiants) identified by the point $z/|z|$ on the unit circle $\mathbb{U}$ (see Eq. (32)). To calculate the argument of z, consider the following relations between algebraic form of z and the trigonometric form

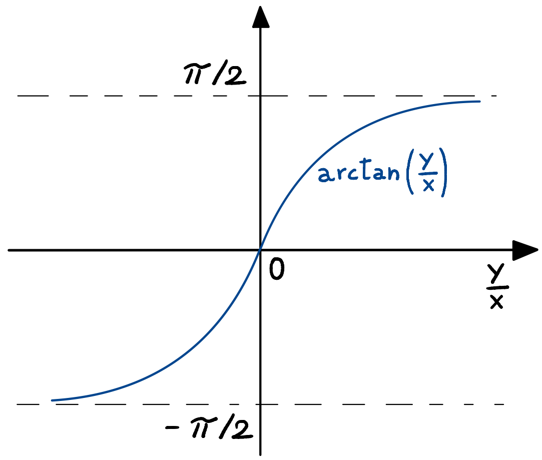
$$x = |z|\cos(\vartheta) \qquad y = |z|\sin(\vartheta) \tag{49}$$

The ratio y/x coincides with the tangent of ϑ

$$\tan(\vartheta) = \frac{y}{x} \tag{50}$$

How do we extract the angle ϑ from the previous equation? One possibility is to use the inverse of the tangent function, i.e., $\arctan(\cdot)$, and write

$$\vartheta = \arctan\left(\frac{y}{x}\right) \tag{51}$$

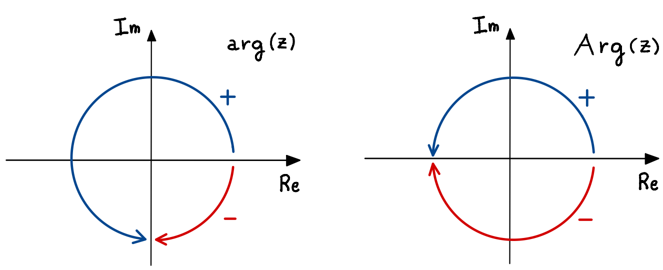


The problem with this simple approach is that function $\arctan(x)$ is defined only in the open interval $]-\pi/2, \pi/2[$. Hence, the expression (51) can be used only to compute the argument of complex number with strictly positive real part[4] (first and fourth quadrants of the complex plane).

To compute the argument of arbitrary complex number $x = x + iy$ we need to shift $\arctan(y/x)$ by π if the real part x is negative

$$\arg(z) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & x > 0 \\ \dfrac{\pi}{2}\mathrm{sign}(y) & x = 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & x < 0 \end{cases} \tag{52}$$

With this definition $\vartheta = \arg(z)$ is unique for all $z \in \mathbb{C}$ and it ranges in $[-\pi/2, 3\pi/2[$.



Alternatively, we can define the argument as (note that here we use capitalized $\mathrm{Arg}(\cdot)$ to distinguish it from (52))

$$\mathrm{Arg}(z) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & x > 0 \\ \dfrac{\pi}{2}\mathrm{sign}(y) & x = 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & x < 0, y \geq 0 \\ \arctan\left(\dfrac{y}{x}\right) - \pi & x < 0, y < 0 \end{cases} \tag{53}$$

With this definition $\vartheta = \mathrm{Arg}(z)$ is unique for all $z \in \mathbb{C}$ and it ranges in $[-\pi, \pi[$.

[4] Complex numbers with argument $\vartheta \in]-\pi/2, \pi/2[$ are either in first quadrant ($\vartheta \in [0, \pi/2[$) or in the fourth quadrant ($\vartheta \in]-\pi/2, 0]$) of the complex plane.

Remark: If we shift the argument of a complex number by $2k\pi$ ($k \in \mathbb{Z}$, the number is not going to change. Hence, the following complex numbers

$$z = 3e^{i\pi/3} \qquad z = 3e^{13i\pi/3} \qquad z = 3e^{-5i\pi/3} \tag{54}$$

are actually the same complex number. This is due to the 2π-periodicity of the circular functions defining the complex exponential (42).

Integer powers of a complex number (De Moivre's formula). Consider a complex number z expressed in a polar form

$$z = |z|e^{i\vartheta}, \tag{55}$$

where $|z|$ is the modulus of z and ϑ denotes its argument. By multiplying z recursively by itself we obtain

$$z^2 = |z|^2 e^{2i\vartheta}, \qquad z^3 = |z|^3 e^{3i\vartheta}, \quad \ldots. \tag{56}$$

Similarly,

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{|z|}{|z|^2}e^{-i\vartheta} = \frac{1}{|z|}e^{-i\vartheta} = |z|^{-1}e^{-i\vartheta} \tag{57}$$

By multiplying $1/z$ recursively by itself we obtain

$$z^{-2} = |z|^{-2}e^{-2i\vartheta}, \qquad z^{-3} = |z|^{-3}e^{-3i\vartheta}, \quad \ldots. \tag{58}$$

Therefore we proved the following Theorem.

Theorem 3 (De Moivre's formula)**.** Let z be any complex number with modulus $|z|$ and argument ϑ. Then

$$z^n = |z|^n e^{in\vartheta} \qquad \forall n \in \mathbb{Z}. \tag{59}$$

Remark: The powers of a complex number complex are points on a spiral in the complex plane. In fact, that the parametric form of a spiral in the Cartesian plane is

$$x(t) = a^t \cos(bt) \qquad y(t) = a^t \sin(bt), \tag{60}$$

where t is the spiral parameter, and a, b are fixed real numbers. These equations coincide with the real and imaginary parts of the powers of z. In fact,

$$\mathrm{Re}(z^n) = |z|^n \cos(n\vartheta) \qquad \mathrm{Im}(z^n) = |z|^n \sin(n\vartheta) \qquad n \in \mathbb{Z}. \tag{61}$$