Lecture 4: Matrices and vectors

A matrix is a rectangular table with entries arranged in rows and columns. The entries can be numbers, functions, operators, matrices, symbols, etc. For example, the following matrix is a 2×3 matrix (2 rows and 3 columns) with real entries

$$A = \begin{bmatrix} 1 & \pi & 2 \\ -\pi & 1 & 0 \end{bmatrix} \tag{1}$$

Similarly,

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{2}$$

is a matrix of trigonometric functions known as *rotation matrix*[1]. In general, an $m \times n$ matrix with entries in some set V has the form

$$A = \begin{array}{cc} \begin{array}{ccccc} & & j\text{-th column} & & \\ & & \downarrow & & \end{array} & \\ \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & \leftarrow i\text{-th row} \end{array} \tag{3}$$

Denote the set of $m \times n$ matrices with entries in V as $M_{m\times n}(V)$. For example, we have:

- $M_{m\times n}(\mathbb{R})$: set of $m \times n$ matrices with real entries

- $M_{m\times n}(\mathbb{C})$: set of $m \times n$ matrices with complex entries

- $M_{m\times n}(C_0([0, 2\pi]))$: set of $m\times n$ matrices with entries in the space continuous functions defined on the interval $[0, 2\pi]$. An element of this set for $n = m = 2$ is the matrix defined in (2), i.e., $R(\theta) \in M_{2\times 2}(C_0([0, 2\pi]))$. Indeed, the entries of $R(\theta)$ are continuous functions in $[0, 2\pi]$.

Example (plotting functions and surfaces): Let us provide a simple example of how vectors and matrices can be used to plot one-dimensional and two-dimensional functions. To this end, consider

$$y = \sin(x) + 2 \qquad x \in [0, 2\pi]. \tag{4}$$

We are interested in plotting this function "point-by-point", i.e., map a set of points $\{x_1, \ldots, x_n\}$ to $y_i = \sin(x_i) + 2$ $(i = 1, \ldots, n)$ one by one. In particular, we choose the set of evenly-spaced points

$$x_{i+1} = \frac{2\pi}{n-1} i \qquad i = 0, \ldots, n-1 \tag{5}$$

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[1] The rotation matrix (2) defines rigid rotations of the Cartesian plane by an angle θ.

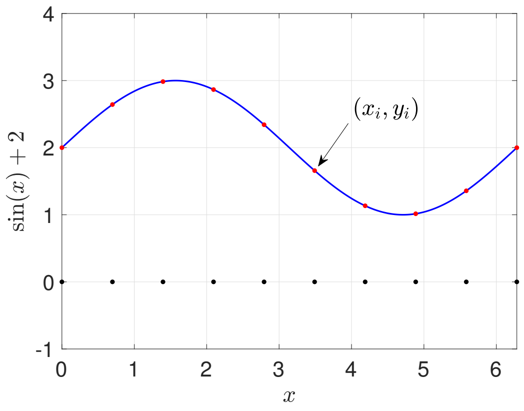
We collect all these points into a matrix with one row and n columns, i.e., a *row vector*

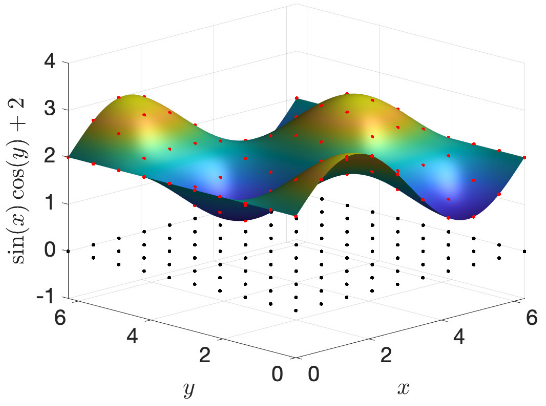
$$x = \begin{bmatrix} 0 & \dfrac{2\pi}{n-1} & \dfrac{4\pi}{n-1} & \cdots & 2\pi \end{bmatrix}. \tag{6}$$

We can then map each entry of the vector x into the corresponding entry of another row vector y as

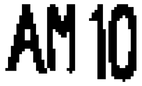
$$y_i = \sin(x_i) + 2 \qquad i = 1, \ldots, n. \tag{7}$$

Hereafter we show the results of this procedure for $n = 10$ in one and two dimensions.





Example (Matrices representing images): The following 50 × 50 black and white image "AM 10"



is represented by the matrix (1 means white, 0 means black)

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1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   0   0   0   1   1   1   1   1   1   1   0   0   1   1   1   1   0   0   0   1   1   1   1   1   1   1   1   0   0   1   1   1   1   1   0   0   0   0   1   1   1   1   1
1   1   1   1   1   1   1   0   0   0   1   1   1   1   1   1   0   0   0   0   1   1   1   0   0   0   1   1   1   1   1   1   1   1   0   0   1   1   1   1   0   0   0   0   0   0   0   1   1   1
1   1   1   1   1   1   1   0   0   0   1   1   1   1   1   1   0   0   0   0   1   1   1   0   0   0   1   1   1   1   1   1   1   0   0   0   1   1   1   0   0   0   0   0   0   0   0   1   1   1
1   1   1   1   1   1   1   0   0   0   0   1   1   1   1   1   0   0   0   0   1   1   1   0   0   0   1   1   1   1   1   1   0   0   0   0   1   1   1   0   0   0   0   0   0   0   0   1   1   1
1   1   1   1   1   1   0   0   0   0   0   1   1   1   1   1   0   0   0   0   0   1   0   0   0   0   1   1   1   1   1   1   0   0   0   0   1   1   1   0   0   0   0   1   0   0   0   0   1   1
1   1   1   1   1   1   0   0   0   0   0   1   1   1   1   1   0   0   0   0   0   1   0   0   0   0   1   1   1   1   1   0   0   0   0   0   1   1   1   0   0   0   1   1   1   0   0   0   1   1
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1   1   1   1   1   0   0   0   0   0   0   0   1   1   1   1   0   0   0   0   0   0   0   0   0   0   1   1   1   1   1   1   1   1   0   0   1   1   0   0   0   1   1   1   1   0   0   0   1   1
1   1   1   1   1   0   0   0   1   1   0   0   1   1   1   1   0   0   1   0   0   0   0   0   0   0   1   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   1   0   0   1   1
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1   1   1   1   0   0   0   0   1   1   0   0   0   1   1   1   0   0   1   1   1   0   0   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   1   0   0   1   1
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1   1   1   1   0   0   0   0   0   0   0   0   0   1   1   1   0   0   1   1   1   1   1   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   1   0   0   1   1
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1   1   1   0   0   0   0   1   1   1   1   0   0   0   1   1   0   0   0   1   1   1   1   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   0   0   0   1   1
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1   1   0   0   0   0   1   1   1   1   1   0   0   0   1   1   0   0   0   1   1   1   1   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   0   0   0   1   1
1   1   0   0   0   0   1   1   1   1   1   0   0   0   0   1   0   0   0   1   1   1   1   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   0   0   0   1   1
1   1   0   0   0   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   1   0   0   1   1   1   1   0   0   0   1   1
1   1   0   0   0   1   1   1   1   1   1   1   0   0   0   1   0   0   0   1   1   1   1   1   0   0   1   1   1   1   1   1   1   1   0   0   0   1   1   0   0   0   1   1   0   0   0   0   1   1
1   0   0   0   0   1   1   1   1   1   1   1   0   0   0   0   1   0   0   1   1   1   1   0   0   0   1   1   1   1   1   1   1   1   0   0   0   1   1   0   0   0   0   0   0   0   0   0   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   0   0   0   1   1   1   1   1   1   1   1   0   1   1   1   1   1   1   1   1   1   0   0   0   1   1   0   0   0   0   0   0   0   0   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   0   0   0   1   1   0   0   0   0   0   0   0   0   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   0   0   0   1   1   1   0   0   0   0   0   0   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   0   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
```

Addition between matrices. It makes sense to define addition between matrices with the same number of rows and and the same number of columns. To this end, let A and B are two $n \times m$ matrices

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \tag{8}$$

We define

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \tag{9}$$

In this way $A+B$ is still an $m \times n$ matrix, i.e., the set of $n \times m$ matrices is closed under the addition operation defined in (9).

Example: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad \Rightarrow \quad A + B = \begin{bmatrix} 1 & 0 & 4 \\ 5 & 3 & 5 \end{bmatrix}. \tag{10}$$

We also define the product between a matrix and number c (real or complex) as

$$cA = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}. \tag{11}$$

Clearly cA is a $m \times n$ matrix.

Examples:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 3 \\ 4 & 1 & 2 \end{bmatrix} & &\Rightarrow \quad 3A = \begin{bmatrix} 3 & 0 & 9 \\ 12 & 3 & 6 \end{bmatrix}. \\ B &= \begin{bmatrix} i & 0 & 3+2i \\ 1+i & i & 2 \\ 1 & 0 & 6i \end{bmatrix} & &\Rightarrow \quad iB = \begin{bmatrix} -1 & 0 & -2+3i \\ -1+i & -1 & 2i \\ i & 0 & -6 \end{bmatrix}. \end{aligned} \tag{12}$$

It is clear that the neutral element for the addition operation (9) is the *zero matrix*

$$0_{n\times m} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}. \tag{13}$$

In fact, for any $m \times n$ matrix A we have $A + 0_{m\times n} = A$. The *opposite* of the matrix A is the matrix[2]

$$-A = \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{bmatrix}. \tag{14}$$

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[2] The opposite of a matrix A is, by definition, the element B such that $A + B = 0_{m\times n}$.

Vector space of matrices. The addition and multiplication by a number operations defined in (9) and (11) satisfy the following properties

1. $A + B = B + A$ (matrix addition is commutative)
2. $(A + B) + C = A + (B + C)$ (matrix addition is associative)
3. $A + 0_{m\times n} = 0_{m\times n} + A = A$ (additive neutral, i.e., the zero matrix)
4. $A - A = 0_{m\times n}$ (opposite matrix $-A$)
5. $c(A + B) = cA + cB$ $\quad c \in \mathbb{R}$ (or $\mathbb{C}$)
6. $(a + b)A = aA + bA$ $\quad a, b \in \mathbb{R}$ (or $\mathbb{C}$)
7. $(ab)A = a(bA)$ $\quad c \in \mathbb{R}$ (or $\mathbb{C}$)

In technical terms, we say that the space of $n \times m$ matrices over the field of complex numbers forms a *vector space*. More generally, any set in which we define an addition operation "+" and a multiplication by $c \in \mathbb{C}$ satisfying properties 1-7 listed above forms is a *vector space over* $\mathbb{C}$. The set of matrices with positive real entries is *not* a vector space since the opposite of a matrix with positive entries is not a matrix with positive entries.

The elements of a vector space are called *vectors*. Hence, a matrix is a vector in the vector space of matrices. A function $f(x) = \sin(x)^2$ is a vector in the vector space continuous functions from $\mathbb{R}$ into $\mathbb{R}$.

Matrix multiplication. Let us consider two matrices A and B and suppose that the number of columns of A (say p) coincides with the number of rows of B

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{p1} & \cdots & b_{pn} \end{bmatrix}. \tag{15}$$

The (standard) *matrix product* between A and B is defined as

$$(AB)_{ij} = a_{i1}b_{1j} + \cdots + a_{ip}b_{pj}, \qquad i = 1, \ldots, m, \quad i = 1, \ldots, n. \tag{16}$$

Here, $(AB)_{ij}$ denotes the (ij)-th entry of the matrix $m \times n$ matrix AB. Note that if the matrix A has size $m \times p$ and the matrix B has size $p \times n$ then the matrix AB defined in (16) has size $m \times n$.

The matrix product (16) corresponds to the so-called *row-column rule* in which the entries of the i-th row of the matrix A are multiplied by the entries j-the column of B and the results of all these multiplications are summed up to obtain the ij-entry of AB

$$\underbrace{\begin{array}{cc} & \begin{array}{ccc} & j & \end{array} \\ i & \begin{bmatrix} & \vdots & & \\ \cdots & \square & & \\ & & & \end{bmatrix} \end{array}}_{AB} = \underbrace{\begin{array}{cc} & \\ i & \begin{bmatrix} [\cdots & \cdots & \cdots & \cdots] \\ & & & \end{bmatrix} \end{array}}_{A} \underbrace{\begin{array}{c} j \\ \begin{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \end{bmatrix} \end{array}}_{B}.$$

Example: Consider the two matrices

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 5 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 1 & -3 \end{bmatrix}. \tag{17}$$

The matrix product AB is well-defined and it is computed as follows:

$$AB = \begin{bmatrix} (1+6+2) & (-1-6-6) \\ (1+10-1) & (-1-10+3) \end{bmatrix} = \begin{bmatrix} 9 & -13 \\ 10 & -8 \end{bmatrix}. \tag{18}$$

Similarly, the matrix product BA in this case is well-defined[3] and it corresponds to the 3×3 matrix

$$BA = \begin{bmatrix} 0 & -2 & 3 \\ 0 & -4 & 6 \\ -2 & -12 & 5 \end{bmatrix}. \tag{19}$$

Remark: For square matrices A and B (i.e., matrices with size $m = p = n$) both products AB and BA are well-defined and they yield $n \times n$ matrices. However, the matrix product is (in general) not commutative, i.e., $AB \neq BA$. For example

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{20}$$

do not commute. In fact, we have

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \\ BA &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 7 & -3 \end{bmatrix}. \end{aligned} \tag{21}$$

The matrix $C = AB - BA$ is called *matrix commutator* of A and B and it often denoted by $C = [A, B]$. If $AB = BA$ then we say that A and B commute. If A and B commute then the commutator $[A, B]$ is the necessarily the zero matrix.

Remark: A very important example of matrix product is the so-called *matrix-vector* product, in which a $m \times n$ matrix A is multiplied by a column vector[4] with n entries

$$Ax = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}. \tag{22}$$

Clearly, Ax is a column vector with entries

$$(Ax)_i = a_{i1}x_1 + \cdots + a_{in}x_n \qquad i = 1, \ldots m. \tag{23}$$

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[3] More generally, if $A \in M_{n\times m}$ and $B \in M_{m\times n}$ then $AB \in M_{n\times n}$ and $BA \in M_{m\times m}$.

[4] A column vector with n entries is a $n \times 1$ matrix.

Remark: The neutral element for the multiplication operation (16) is called *identity matrix*

$$I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in M_{n\times n}(\mathbb{R}) \tag{24}$$

The identity matrix is a square matrix with ones along the main diagonal and zeros everywhere else. If A is a $m \times n$ matrix then

$$I_m A = A I_n = A \tag{25}$$

Theorem 1 (Properties of the matrix product)**.** Let A, B and C three matrices for which the following products and sums are well-defined. Then:

1. $A(BC) = (AB)C$ (matrix multiplication is associative),
2. $A(B + C) = AB + AC$ (left distributive property),
3. $(A + B)C = AC + BC$ (right distributive property),
4. $c(AB) = A(cB)$, $c \in \mathbb{C}$.

Proof. Properties 1 to 4 can be proved simply by using the definition (9), (11) and (16). Let us prove property 2. To this end, let $B, C \in M_{n\times m}$ and $A \in M_{p\times n}$ so that the matrix multiplication in property 2 is well-defined. The ij entry of the matrix $A(B + C)$ can be written as

$$(A(B+C))_{ij} = \sum_{p=1}^{n} a_{ip}(b_{ij} + c_{pj}) = \sum_{p=1}^{n} a_{ip}b_{ij} + \sum_{p=1}^{n} a_{ip}c_{ij} = (AB)_{ij} + (AC)_{ij}. \tag{26}$$

□

Remark: Consider an arbitrary square matrix A and a positive integer p. The p-th power of A is the matrix

$$A^p = \underbrace{AA\cdots A}_{p \text{ times}} \qquad \text{(matrix power).} \tag{27}$$

For example, the square of the matrix A defined in equation (20) is

$$A^2 = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -3 & 7 \end{bmatrix}. \tag{28}$$

Remark: It is possible to define other types of matrix products, e.g., the Kronecker product "⊗" or the Hadamard product "∘". These types of products are different from the matrix product (16), and they satisfy different properties. For example, the Hadamard product between the matrices

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

is defined as

$$A \circ B = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{1n}b_{1n} \\ \vdots & & \vdots \\ a_{m1}b_{m1} & \cdots & a_{mn}b_{mn} \end{bmatrix} \qquad \text{(Hadamard product).} \tag{29}$$

and it is clearly commutative[5], i.e., $A \circ B = B \circ A$. On the other hand, given two matrices $A \in M_{n\times m}$ and $B \in M_{p\times q}$ their Kronecker product is defined as defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \qquad \text{(Kronecker product).} \tag{30}$$

Note that that $A \otimes B$ is a block matrix of size $np \times mq$. In fact, each entry of $A \otimes B$ is a matrix of size $p \times q$.

Transpose of a matrix. The transpose of the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \tag{31}$$

is the matrix obtained by switching the row and column indices of A, i.e.,

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}. \tag{32}$$

For example,

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -2 & 4 & 1 \end{bmatrix} \quad \Leftrightarrow \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 1 & 4 \\ 3 & 1 \end{bmatrix}. \tag{33}$$

Theorem 2 (Properties of transpose matrix)**.** Let A and B two matrices for which the following operations are well-defined. Then:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(cA)^T = cA^T \qquad c \in \mathbb{C}$

---

[5] Recall that the standard matrix product between two square matrices is (in general) not commutative, i.e, $AB \neq BA$.

Proof. Let us prove property 3. To this end, let $A \in M_{n\times m}$ and $B \in M_{m\times p}$. The ij entry of the matrix AB is (see equation (16))

$$(AB)_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj} \tag{34}$$

To obtain the ij entry of $(AB)^T$ we simply need to switch i and j. This yields,

$$\left((AB)^T\right)_{ij} = \sum_{k=1}^{m} a_{jk}b_{ki} = \sum_{k=1}^{m} \left(B^T\right)_{ik}\left(A^T\right)_{kj} = \left(B^TA^T\right)_{ij} \tag{35}$$

All other properties can be proved in a similar way.

□

Remark: Let A, B, C and D four matrices such that the product $ABCD$ is well-defined. Then

$$(ABCD)^T = D^TC^TB^TA^T \tag{36}$$

In fact, by applying property 3 in Theorem 2 recursively we have

$$(ABCD)^T = (CD)^T(AB)^T = D^TC^TB^TA^T \tag{37}$$

Remark (Conjugate transpose): For matrices with complex entries we can also define the conjugate transpose as

$$A^H = \left(A^T\right)^* \tag{38}$$

The conjugate transpose of a matrix A has entries

$$a^H_{ij} = a^*_{ji}. \tag{39}$$

Symmetric and skew-symmetric matrices. Let $A \in M_{n\times n}$ be a square matrix[6].

If $A = A^T$ then we say that A is *symmetric*.

If $A = -A^T$ then we say that A is *skew-symmetric* (or *anti-symmetric*).

Examples of symmetric and skew-symmetric matrices are

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 3 & -3 & 5 \\ 1 & 5 & 0 \end{bmatrix} \quad \text{(symmetric)}, \qquad B = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} \quad \text{(skew-symmetric)}. \tag{40}$$

By definition, the entries of a symmetric matrix A satisfy $a_{ij} = a_{ji}$. Similarly, the entries of a skew-symmetric matrix satisfy $a_{ij} = -a_{ji}$. Note that this implies that the diagonal entries of a skew symmetric matrix are necessarily zero

$$a_{ii} = -a_{ii} \quad \Rightarrow \quad a_{ii} = 0. \tag{41}$$

---

[6] The definition of symmetric and skew-symmetric matrices makes sense only for square matrices. In fact, the statements $A = A^T$ and $A = -A^T$ are legitimate only for square matrix. Otherwise we are saying that, e.g., a 3×2 matrix equals a 2×3 matrix.

Any square matrix $A \in M_{n\times n}$ can be decomposed into a sum of a symmetric matrix and a skew-symmetric matrix as follows

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew-symmetric}} . \tag{42}$$

The following result holds for arbitrary rectangular matrices.

Theorem 3. Let $A \in M_{n\times m}$ be an arbitrary $n \times m$ matrix. Then AA^T is a $n \times n$ symmetric matrix and $A^T A$ is a $m \times m$ symmetric matrix.

The proof is left as exercise.

Remark: If A is a $n \times n$ square matrix then $A^T A$ and AA^T are both $n \times n$ symmetric matrices. In general, $A^T A \neq AA^T$. However, if A is symmetric then $AA^T = A^T A$ (show it!).

Matrix inverse. Let $A \in M_{n\times n}$ be a square matrix. We say that A is *invertible* if there exists a $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I_n \qquad A^{-1}A = I_n. \tag{43}$$

where I_n is the identity matrix (24).

Theorem 4 (Uniqueness of the inverse matrix)**.** The matrix A^{-1} satisfying (43) is unique.

Proof. Suppose that there are two matrices B_1 and B_2 such that

$$AB_1 = I_n, \quad B_1A = I_n \qquad \text{and} \qquad AB_2 = I_n, \quad B_2A = I_n. \tag{44}$$

Then

$$B_2 = B_2I_n = B_2(AB_1) = (B_2A)B_1 = B_1, \tag{45}$$

i.e., $B_2 = B_1$. This means that for any matrix A, the inverse is unique (if it exists).

□

Hence, if A is invertible[7] then there exists a unique matrix A^{-1} that commutes with A such that the matrix product between A and A^{-1} yields the identity matrix (24).

Theorem 5 (Properties of the inverse matrix)**.** Let A and B be two $n \times n$ invertible matrices. Then

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $\left(A^T\right)^{-1} = (A^{-1})^T$

Proof. Let us prove properties 1, 2 and 3.

1. Let C be the inverse of A^{-1}. Then

$$A^{-1}C = I_n \qquad CA^{-1} = I_n. \tag{46}$$

Theorem 3 says that there exists only one matrix that satisfies (46), and that matrix is A. Thus, the inverse of A^{-1} is A.

---

[7] We will derive conditions for the invertibility of a matrix A in subsequent lecture notes. As we will see, not every square matrix admits an inverse.

2. The following identities
$$I_n = AB(B^{-1}A^{-1}), \qquad I_n = (B^{-1}A^{-1})AB \tag{47}$$
imply that the inverse of the matrix AB is $B^{-1}A^{-1}$.

3. Consider
$$I_n = (AA^{-1})^T = (A^{-1})^T A^T \qquad I_n = (A^{-1}A)^T = A^T(A^{-1})^T. \tag{48}$$
Therefore the inverse of A^T, i.e. $(A^T)^{-1}$, is equal to $(A^{-1})^T$.

$\square$

Orthogonal and unitary matrices. Let $A \in M_{n\times n}$ be a square matrix with real entries. We say that A is an *orthogonal matrix* [8] if
$$AA^T = A^T A = I_n. \tag{49}$$
Clearly, if A is an orthogonal matrix then (by using the definition of the inverse and its uniqueness)
$$A^T = A^{-1}. \tag{50}$$

Moreover, if A is an orthogonal matrix then the commutator
$$[A, A^T] = AA^T - A^T A = I_n - I_n = 0_{M_{n\times n}}. \tag{51}$$
If the matrix A has complex entries then we say that A is a *unitary matrix* if
$$AA^H = A^H A = I_n \tag{52}$$
where A^H is the conjugate transpose of A.

Linear systems of equations. Consider the linear system of equations (m equations in n unknowns)
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \qquad\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{53}$$
Upon definition of
$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \tag{54}$$
we can write (53) in a matrix-vector product form as
$$Ax = b. \tag{55}$$
In the particular case where $m = n$ (number equations equals the number of unknowns) we have that if the matrix A is invertible then the system (53) admit the unique solution[9]
$$x = A^{-1}b \tag{56}$$
As we shall see in the next lecture, there is no need to compute the inverse matrix A^{-1} to solve the linear system (53).

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[8] As we will see, the reason why we call the matrix A satisfying (49) an orthogonal matrix follows from the fact that the rows (or the columns) of such matrix are orthonormal relative to standard "dot product" in $\mathbb{R}^n$.

[9] By applying A^{-1} to both sides of (55) we obtain $\underbrace{A^{-1}A}_{I_n}x = A^{-1}b$, i.e., $x = A^{-1}b$.