Lecture 5: Linear equations

An equation in n variables is *linear* if it can be written in the form

$$a_n x_n + \ldots + a_1 x_1 = b. \tag{1}$$

The numbers $\{a_1, \ldots, a_n\}$ are the *coefficients* of the equation, while b is usually called *constant term.*

The variables x_j and the constant term b can be elements of rather general vector spaces. For example, x_j can be vectors in $\mathbb{R}^n$, $n \times m$ matrices with real entries, or real-valued continuous functions, while a_i are usually real or complex numbers[1].

Examples: Let us provide a few simple examples of linear equations in the space $\mathbb{R}^n$ for $n = 2$ and $n = 3$. The elements of $\mathbb{R}^n$ are n-tuples of real numbers of the form

$$x = (x_1, \ldots, x_n) \qquad x_i \in \mathbb{R}. \tag{2}$$

In a matrix setting, x can be represented as a row vector or as a column vector

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}. \tag{3}$$

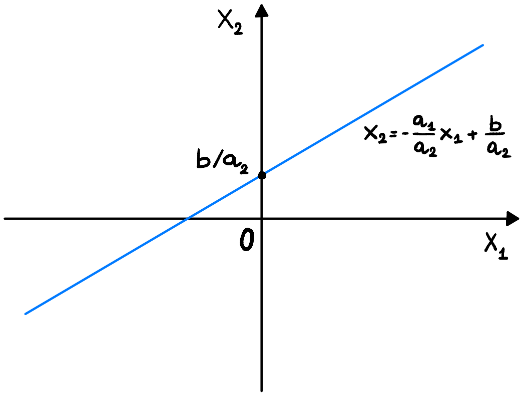
(a) The linear equation

$$a_1 x_1 + a_2 x_2 = b \qquad a_1, a_2, b \in \mathbb{R} \tag{4}$$

represents a *line* in $\mathbb{R}^2$. In fact, if $a_2 \neq 0$ then we can express x_2 in terms of x_1 as

$$x_2 = -\frac{a_1}{a_2} x_1 + \frac{b}{a_2}. \tag{5}$$

The graph x_2 versus x_1 is, e.g.,



If $a_2 = 0$ and $a_1 \neq 0$ we obtain the vertical line $x_1 = b/a_1$. Lastly, if $a_1 = a_2 = 0$ then we necessarily have $b = 0$ and the linear equation reduces to $0 = 0$, which is uninformative.

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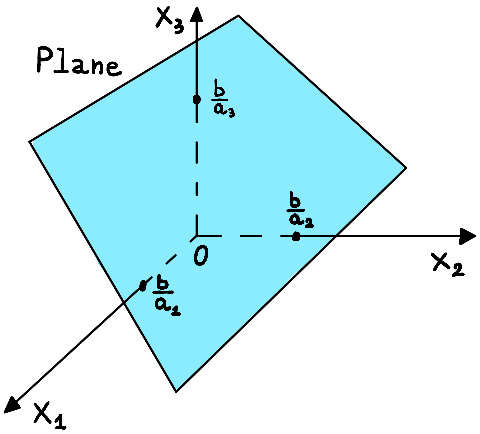
[1] The vast majority of vector spaces are constructed over the field $\mathbb{R}$ or $\mathbb{C}$.

(b) The linear equation

$$a_3x_3 + a_2x_2 + a_1x_1 = b \qquad a_i, b \in \mathbb{R} \tag{6}$$

represents a *plane* in $\mathbb{R}^3$. Such a plane is a two-dimensional surface embedded in three dimensional space, which can be sketched as follows



This plane can be also expressed as a linear combination (linear equation) of two 3D vectors lying on the plane, plus a constant 3D vector.

(c) The following linear equation represents a so-called *hyper-plane* in $\mathbb{R}^n$ $(n \geq 4)$.

$$a_nx_n + \cdots + a_1x_1 = b \qquad a_i, b \in \mathbb{R} \tag{7}$$

Systems of linear equations. A system of m linear equations of the form (1) can be written as

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_n \end{cases} \tag{8}$$

For example,

$$\begin{cases} 3x_1 + 2x_2 = 1 \\ x_1 - 5x_2 = 0 \end{cases} \qquad \text{2 equations in 2 variables}$$

$$\begin{cases} 5x_1 - x_3 = 3 \\ x_1 + 2x_2 - 8x_3 = 5 \end{cases} \qquad \text{2 equations in 3 variables}$$

A solution to the linear system (8) is a set n variables $(x_1, \ldots, x_n)$ satisfying all equations in (8). In general, linear systems can have

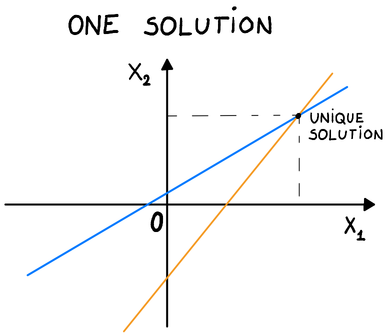
1. Exactly one solution
2. No solution
3. Infinite solutions

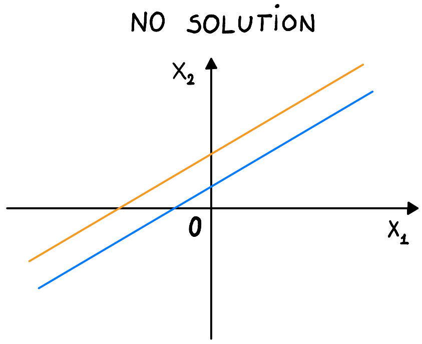
Geometric interpretation:

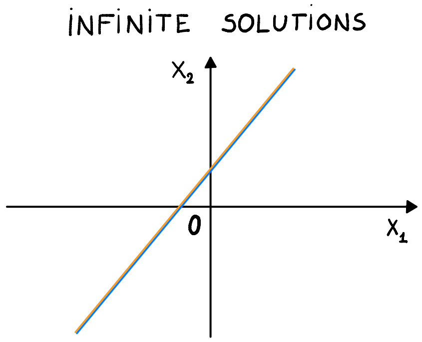
- We have seen that a linear equation in $\mathbb{R}^2$ defines a line in the Cartesian plane. Hence, the following system of two equations in $\mathbb{R}^2$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{9}$$

defines two lines. Such lines can intersect at one point (unique solution), can be parallel (no solutions) or they can be superimposed (infinite solutions).


ONE SOLUTION

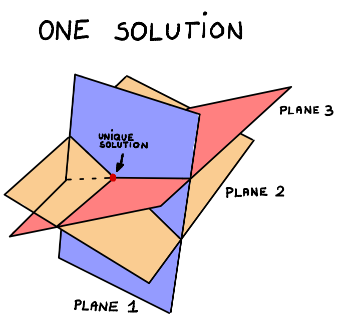

NO SOLUTION

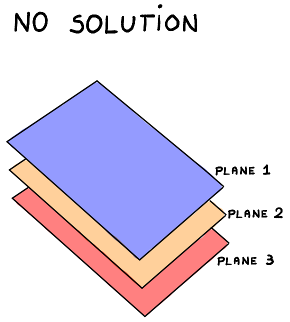

INFINITE SOLUTIONS

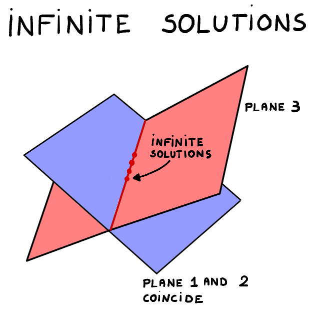
- We have seen that a linear equation in $\mathbb{R}^3$ defines a plane in the three-dimensional space. Hence, the following three equations in $\mathbb{R}^3$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_2 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \tag{10}$$

define three planes. Such planes can intersect at one point (unique solution), can be parallel and distinct (no solution if just two planes are parallel), or they can intersect along one line (infinite solutions, one-dimensional set), or even be the same plane (infinite solutions, two-dimensional set).


ONE SOLUTION


NO SOLUTION


INFINITE SOLUTIONS

Remark: A linear system of m equations in n variables can be written in a compact matrix-vector form as

$$Ax = b \tag{11}$$

where

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}. \tag{12}$$

Solving a linear system of equations. Let us begin with the the following simple example of a system of 2 linear equations in 2 unknowns

$$\begin{cases} x_1 + x_2 = 3 \\ x_1 - 2x_2 = 1 \end{cases} \tag{13}$$

Clearly, we can express x_1 in terms of x_2 by using the second equation,i.e.,

$$x_1 = 1 + 2x_2 \tag{14}$$

and then substitute this result into the first equation to obtain

$$1 + 2x_2 + x_2 = 3 \quad \Rightarrow \quad x_2 = \frac{2}{3} \tag{15}$$

$$x_1 = 1 + 2\left(\frac{2}{3}\right) \quad \Rightarrow \quad x_1 = \frac{7}{3} \tag{16}$$

Note that $x_1 = 2/3$ and $x_2 = 7/3$ satisfy (13). The method we just described, is not very efficient for linear systems in higher dimensions, e.g.,

$$\begin{cases} x_1 + x_2 + x_3 + x_4 - 3x_5 - = 1 \\ x_1 - x_2 + x_3 - x_4 - 12x_5 = 2 \\ 3x_1 - 3x_2 + x_3 + x_4 + x_5 = -2 \\ -x_1 + 2x_2 + x_3 + x_4 + -4x_5 = -2 \\ -4x_1 - x_2 + x_3 + x_4 + x_5 = -2 \end{cases}$$

A more effective method relies on transforming a linear system into an *equivalent* one, i.e., a systems with the same solutions, that is easier to solve. The key observation is the following:

The solution of a linear system does not change if we replace one equation with a linear combination of that equation and others in the system (we will see why!).

Is this true? Let us verify the statement in the simplest possible setting, i.e., for the 2 × 2 linear system

$$\begin{cases} 2x_1 + x_2 = 1 \\ x_1 + x_2 = 0 \end{cases} . \tag{17}$$

This system can be written in a matrix-vector form as

$$\underbrace{\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{b}. \tag{18}$$

The solution is clearly $x_1 = 1$ and $x_2 = -1$. Let us now replace the second equation in (17), i.e., $x_1 + x_2 = 0$, with the first equation multiplied by 2 plus the second. This yields

$$\begin{cases}2x_1 + x_2 = 1\\ 5x_1 + 3x_2 = 2\end{cases} \tag{19}$$

which still has the unique solution $x_1 = 1$ and $x_2 = -1$. So the statement seems to be true.

If we replace the second equation in (17) by the second equation multiplied by 2 itself minus the first equation we can *eliminate* the variable x_1 to obtain

$$\begin{cases}2x_1 + x_2 = 1\\ x_2 = -1\end{cases} \tag{20}$$

This system can be written in a matrix-vector form as

$$\underbrace{\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}}_{A_1}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}1\\ -1\end{bmatrix}}_{b_1} \tag{21}$$

The matrix A has an *upper-trianglar* triangular structure which allows us to solve the system by using *backward substitution*, i.e., solving the last equation first and then substituting the result back into into the previous equations.

Remark: Note that the operation we just described, i.e, "subtract the first equation from the second multiplied by 2" can be represented by a *lower-triangular* (invertible) matrix

$$T_1 = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix} \tag{22}$$

In fact, by applying T_1 to equation (18) we obtain equation (21), i.e.,

$$T_1 A x = T_1 b \quad \Rightarrow \quad A_1 x = b_1 \tag{23}$$

This can be verified by a direct calculation

$$T_1 A = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix} = A_1 \qquad T_1 b = \begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix} = b_1. \tag{24}$$

Gauss elimination method and row echelon forms of a matrix. The method we just described to transform a linear system in an "upper triangular" form is known as *Gauss elimination method*, and it can be applied to linear systems with an arbitrary number of linear equations and an arbitrary number of unknowns.

When performing Gaussian elimination is also convenient to interchange the rows of the augmented matrix so that the row with largest (in absolute value) entry acts as a pivot for the elimination step. This procedure is called *Gauss elimination method with pivoting by row.* In general, the following *elementary row operations* performed on the augmented matrix do not change the solution of the associated linear system of equations:

1. multiplication of one row by a non-zero number,
2. addition of one row to another, and
3. interchange two rows.

All these operations can be represented by invertible matrices. This implies that they do not change the solution of the system. In fact, if T is an invertible matrix then

$$Ax = b \quad \Leftrightarrow \quad TAx = Tb. \tag{25}$$

In orther words, $Ax = b$ and $TAx = Tb$ have the same solution. Note that it is possible to transform $TAx = Tb$ back into $Ax = b$ if and only if T is invertible[2]. On the other hand, if T is *not* invertible then

$$Ax = b \quad \Rightarrow \quad TAx = Tb, \qquad \text{but} \qquad TAx = Tb \not\Rightarrow Ax = b. \tag{26}$$

This means that the systems are not equivalent if T is not invertible. Let us clarify why elementary row operations on a matrix can be represented as multiplications by invertible matrices.

Example: Consider the following 2×4 matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ -3 & 1 & 2 & -2 \end{bmatrix} \tag{27}$$

The interchange of the first and the second row is represented by the matrix T_1

$$\begin{bmatrix} -3 & 1 & 2 & -2 \\ 1 & 2 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{T_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 2 & -2 \end{array}\right] \tag{28}$$

Similarly, multiplication of the first row by $-1/3$ is represented by the matrix T_2

$$\begin{bmatrix} 1 & -1/3 & -2/3 & 2/3 \\ 1 & 2 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/3 & 0 \\ 0 & 1 \end{bmatrix}}_{T_2} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{T_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ -3 & 1 & 2 & -2 \end{bmatrix} \tag{29}$$

Finally, the subtraction of the first row from the second one is represented by the matrix T_3

$$\begin{bmatrix} 1 & -1/3 & -2/3 & 2/3 \\ 0 & 4/3 & 5/3 & 1/3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{T_3} \underbrace{\begin{bmatrix} -1/3 & 0 \\ 0 & 1 \end{bmatrix}}_{T_2} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{T_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ -3 & 1 & 2 & -2 \end{bmatrix}. \tag{30}$$

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[2] Just apply T^{-1} to $TAx = Tb$ to obtain $Ax = b$.

The matrices T_1, T_2 and T_3 are all invertible, and therefore their product $T = T_3T_2T_1$ is invertible[3]. The invertibility of T establishes a one-to-one correspondence between the matrix (27) and the matrix at the left hand side of (30).

The matrix (30) is said to be in *row echelon form* A matrix is in row echelon form if:

Whenever two successive rows do not consist entirely of zeros, then the second row starts with a non-zero entry at least one step further to the right than the first row. All the rows consisting entirely of zeros are at the bottom of the matrix. The row echelon form of a matrix is not unique.

Let us now show how to solve a linear system by using Gauss elimination with pivoting by row. To this end, consider the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 2x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 1 \end{cases} \tag{32}$$

This system can be written in a matrix-vector form as

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{b}. \tag{33}$$

Define the following *augmented matrix* associated with (32) (or equivalently (33))

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \tag{34}$$

Note that the augmented matrix is obtained by concatenating the column vector b to the right of the matrix A. As we shall see hereafter, the Gauss elimination method with pivoting by row yields an augmented matrix in row echelon form.

Let us know describe the Gauss elimination method with pivoting by row which will transform the augmented matrix (34) in row echelon form.

1. Pivoting step: We select the equation with the largest absolute value of a_{i1}, i.e., the second equation in (33), and we interchange it with the first to obtain

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

(Augmented matrix of the new system)

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[3] Recall that the inverse of a production of invertible matrices T_1, T_2 and T_3 is invertible and that

$$(T_3T_2T_1)^{-1} = T_1^{-1}T_2^{-1}T_3^{-1}. \tag{31}$$

2. Elimination step: We multiply the first equation by $-1/2$ and add it to the second and the third equation. This yields,

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + x_3 - x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 2 - \frac{1}{2} \\ x_1 + x_2 + x_3 - x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 1 - \frac{1}{2} \end{cases} \Rightarrow \begin{cases} 2x_1 + x_2 + x_3 = 1 \\ \frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{3}{2} \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2} \end{cases}$$

Therefore, we obtain

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ \frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{3}{2} \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 = \frac{1}{2} \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 3/2 \\ 0 & 1/2 & 1/2 & 1/2 \end{array}\right]$$

(Augmented matrix of the new system)

3. Pivoting step: We look for the equation with the maximum absolute value of the coefficient $a_{j2}, (j \geq 2)$. In this case, it is the second equation. Hence, we do not do any permutation.
4. Elimination step: We multiply the second equation by $-1/3$ and we add it to the last one to eliminate x_2

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ \frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{3}{2} \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_2 - \frac{1}{6}x_3 = \frac{1}{2} - \frac{1}{2} \end{cases} \Rightarrow \begin{cases} 2x_1 + x_2 + x_3 = 1 \\ \frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{3}{2} \\ \frac{1}{3}x_3 = 0 \end{cases}$$

Thus, we obtained

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ \frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{3}{2} \\ \frac{1}{3}x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 3/2 \\ 0 & 0 & 1/3 & 0 \end{array}\right] \tag{35}$$

(Augmented matrix in *row echelon form*)

At this point we can now use backward substitution (i.e. solve the system of equations form the bottom to the top). This yields the following unique solution to the system (33)

$$\begin{cases} x_3 = 0 \\ x_2 = \frac{2}{3}\left(\frac{3}{2} - \frac{1}{2}x_3\right) = \frac{2}{3}\left(\frac{3}{2} - \frac{1}{2}(0)\right) = 1 \\ x_1 = \frac{1}{2}(1 - x_2 - x_3) = \frac{1}{2}(1 - 1 - 0) = 0 \end{cases}$$

Remark: For a given system of linear equations, the row echelon forms is *not* unique. In fact there is infinite number of ways by which the augmented matrix of a linear system can be transformed in a row echelon form. For example, if we perform Gauss elimination without pivoting in (33), then we obtain the following row echelon form

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ -3x_2 - x_3 = -3 \\ \dfrac{1}{3}x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -3 \\ 0 & 0 & 1/3 & 0 \end{array}\right] \tag{36}$$

(Augmented matrix in *row echelon form*)

The row echelon forms (35) and (36) are different, but they are both obtained from by apply elementary row operations to the same linear system (33).

Reduced row echelon form. The Gauss elimination method with pivoting by row can be applied to any linear system of equations (e.g., 2 equations in 3 unknowns) to obtain a row echelon form. Once the row echelon form is available, then we can normalize the entries of a certain row by dividing them by the pivot, and then perform *backward elimination* to remove all entries above such pivot. In numerical linear algebra this is known as *Jordan backward elimination.* Let us show how this works. To this end, consider the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + \dfrac{1}{3}x_3 = 1 \\ x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1/3 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(row echelon form)

Multiply the third equation by 1/3 and 1, respectively, and subtract it from the second and first equation, respectively. This yields

$$\begin{cases} x_1 + 2x_2 = 2 \\ x_2 = 1 \\ x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(still in row echelon form)

Finally, multiply the second equation by 2 and subtract it from the first equation to obtain

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(reduced row echelon form)

The augmented matrix of a linear system is in a *reduced row echelon form* if: 1) it is in an echelon form; and 2) in every pivot column, the pivot value is 1 and all other entries are 0. The reduced row echelon form of a matrix or linear system is *unique.*

Example: Consider the augmented matrix in row echelon form we obtained by performing Gauss elimination on (33) without pivoting, i.e.,

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ -3x_2 - x_3 = -3 \\ \dfrac{1}{3}x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -3 \\ 0 & 0 & 1/3 & 0 \end{array}\right]$$

(row echelon form)

To obtain the reduced row echelon form, we first rescale the third equation equation by multiplying it by 3. This yields,

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ -3x_2 - x_3 = -3 \\ x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(row echelon form)

Next, we perform backward elimination of x_3 to obtain

$$\begin{cases} x_1 + 2x_2 = 2 \\ -3x_2 = -3 \\ x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(row echelon form)

At this point, we rescale the second equation by $-1/3$ and use it to eliminate x_2 in the first equation. This yields

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 0 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(reduced row echelon form)

Note that the last column of the reduced-row echelon form is the solution of the system (33).

Example: The following matrices are in a reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example: The following matrices are not in a reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 5 & 7 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Remark: A linear system is said to be *consistent* if admits a solution. A system admits a solution (and therefore it is consistent) if and only if the row echelon form (or the reduced row echelon form) of the augmented matrix has no row of the form:

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & z \end{array}\right], \qquad z \neq 0$$

If the system is consistent then we can have one (unique) solution or infinitely many. An example of a system that is not consistent is the following

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_1 + x_2 - x_3 = 4 \end{cases}$$

This system defines two parallel planes (not intersecting). The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right],$$

and therefore the system is not consistent.

Computation of the inverse matrix. Let $A \in M_{n\times n}$ be an invertible matrix. By definition, the inverse of A is a square matrix denoted as A^{-1} with the following properties

$$AA^{-1} = I_n \qquad A^{-1}A = I_n, \tag{37}$$

where I_n is the $n \times n$ identity matrix. Let h_i be the columns of the matrix A^{-1}, i.e.,

$$A^{-1} = [h_1 \quad h_2 \quad \cdots \quad h_n] \qquad h_i \in M_{n\times 1} \quad i = 1, \dots, n. \tag{38}$$

By definition of matrix-vector product we have

$$AA^{-1} = [Ah_1 \quad Ah_2 \quad \cdots \quad Ah_n]. \tag{39}$$

At this point we define the following column vectors $e_i \in M_{n\times 1}$ $(i = 1, ..., n)$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \qquad \cdots, \qquad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{40}$$

Note that e_i is the i-th column of the identity matrix I_n. With this notation we can write the matrix equation $AA^{-1} = I_n$ as

$$[Ah_1 \quad Ah_2 \quad \cdots \quad Ah_n] = [e_1 \quad e_2 \quad \cdots \quad e_n]. \tag{41}$$

Hence, the n columns of the inverse matrix A^{-1}, i.e., h_1, ..., h_n are solutions to n linear systems

$$Ah_1 = e_1, \qquad Ah_2 = e_2, \qquad \dots, \qquad Ah_n = e_n. \tag{42}$$

To solve these systems we can compute the reduced row echelon form of the following augmented matrices

$$\left[A \,\middle|\, e_1\right], \qquad \left[A \,\middle|\, e_2\right], \qquad \dots, \qquad \left[A \,\middle|\, e_n\right] \tag{43}$$

If A is invertible, then A can be row-reduced to I_n. This means that the reduced row echelon form of the systems (43) is

$$\left[I_n \mid h_1\right], \qquad \left[I_n \mid h_2\right], \qquad \dots, \qquad \left[I_n \mid h_n\right], \tag{44}$$

where h_i is the i-the column of the inverse matrix.

More compactly, we can compute the reduced row echelon form of the matrix

$$\left[A \mid I_n\right] \qquad \text{to obtain} \qquad \left[I_n \mid A^{-1}.\right] \tag{45}$$

Example: Compute the inverse of the following 2×2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{46}$$

We begin by constructing the augmented matrix $[A|I_2]$

$$\left[A \mid I_2\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \tag{47}$$

Then we transform the augmented matrix into row-reduced echelon form as

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_2 : -R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \tag{48}$$

$$\xrightarrow{R_1 : R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]. \quad \text{with } \underbrace{\begin{matrix} -1 & 2 \\ 1 & -1 \end{matrix}}_{A^{-1}} \tag{49}$$

Hence, the inverse of the matrix A defined in (46) is

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}. \tag{50}$$

It is good practice to verify that A^{-1} is indeed the inverse of A. To this end, we just need to check that $AA^{-1} = I_2$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \tag{51}$$

Example: Compute the inverse of the following 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{52}$$

As before,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 : R_3 - R_1]{R_2 : R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : -R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:R_3+2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_3:-R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \xrightarrow{R_2:R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \xrightarrow{R_1:R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]. \tag{53}$$

Therefore, the inverse of the matrix A defined in (52) is

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix}. \tag{54}$$