Lecture 6: Vector spaces

Vector spaces are sets in which we define an addition operation and a multiplication by a scalar satisfying certain number of properties. Let us first give a formal definition of vector space and then provide a few examples. Consider a nonempty set V in which define an addition operation "+" satisfying the following properties[1]:

1. $\forall u, v \in V \ (u+v) \in V$ (V is closed under the addition operation)
2. $\forall u, v \in V \ u+v = v+u$ (addition is commutative)
3. $\forall u, v, w \in V \ (u+v)+w = u+(v+w) \in V$ (addition is associative)
4. $\exists\, 0_V \in V$ such that $u + 0_V = u \quad \forall u \in V$ (additive neutral)
5. $\forall u \in V$, $\exists v \in V$ such that $u + v = 0_V$ (opposite element)

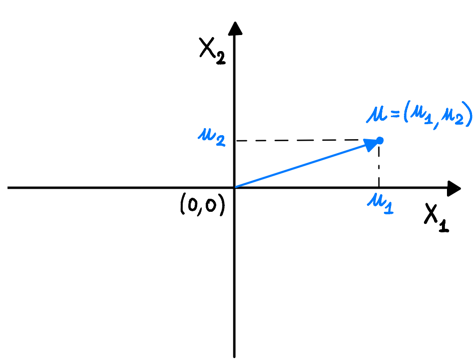
We also define the multiplication operation between an element of the set V and an element of a field K (e.g., $\mathbb{R}$ or $\mathbb{C}$) with the following properties:

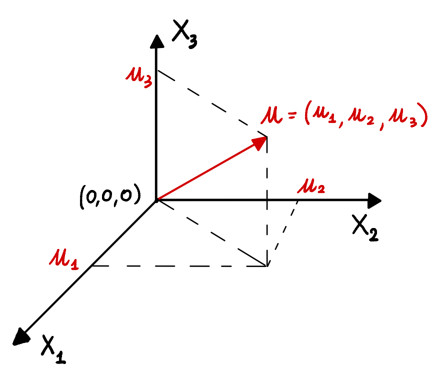
1. $av \in V \quad \forall a \in K, \quad \forall v \in V$
2. $(a+b)v = av + bv \quad \forall a, b \in K, \quad \forall v \in V$
3. $a(v+w) = av + aw \quad \forall a \in K, \quad \forall v, w \in V$
4. $(ab)v = a(bv) \quad \forall a, b \in K, \quad \forall v \in V$
5. $1v = v \quad 1 \in K, \forall v \in V$

Definition (Vector space)**.** A nonempty set V in which we define an addition operation and a multiplication operation satisfying the properties listed above is called *vector space* over K.

Let us provide a few examples of vector spaces over the real or complex numbers.

- The space $\mathbb{R}^n$ (n-tuples of real numbers) with the addition operation defined as $u + v = (u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$ is a vector space over $\mathbb{R}$. The neutral element with respect to the addition operation is $0_{\mathbb{R}^n} = (0, 0, \ldots, 0)$. Here is a simple visualization of a vector u in the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.





[1] A set V satisfying properties 1 to 5 is called "Abelian group".

- $V = M_{m\times n}(\mathbb{R})$, i.e., the set of real $m \times n$ matrices with the addition operation we defined in Lecture 4, is a vector space over $\mathbb{R}$. The neutral element with respect to the addition operation is

$$0_{M_{m\times n}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{(zero matrix).} \tag{1}$$

- $V = M_{n\times m}(\mathbb{C})$ is a vector space over $\mathbb{C}$ and over $\mathbb{R}$.
- $V = \mathbb{P}_n(\mathbb{R})$, i.e., the space of polynomials of degree n with real coefficients, is a vector space over $\mathbb{R}$. An element of $\mathbb{P}_n(\mathbb{R})$ is

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad a_j \in \mathbb{R}, \quad x \in \mathbb{R}. \tag{2}$$

  The addition operation between two polynomials, say $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $q(x) = b_0 + b_1 x + \cdots + b_n x^n$, is defined as

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n, \qquad a_j, b_j \in \mathbb{R}, \quad x \in \mathbb{R}. \tag{3}$$

  The neutral element with respect to the addition operation is the zero polynomial $p(x) = 0$.
- $V = C^{(1)}(\mathbb{R})$ (space of real-valued continuously differentiable functions defined on the real line) is a vector space over $\mathbb{R}$. An element of $C^{(1)}(\mathbb{R})$ is, e.g., $v(x) = e^{-x^2}\sin(x)$. The neutral element with respect to the addition operation is the zero function $v(x) = 0$.
- Then space of linear transformations between two vector spaces V and W is a vector space over $\mathbb{R}$. The elements of such vector space are linear maps $\mathcal{L} : V \to W$.

Vector subspace. Let V be a vector space over a field K. We say that $W \subseteq V$ is a vector subspace of V if

1. $0_V \in W$
2. $u, v \in W \quad \Rightarrow (u + v) \in W$
3. $cu \in W \quad \forall u \in W, \quad \forall c \in K$

Clearly, a vector subspace is itself a vector space. Note that the only condition we need for $W \subseteq V$ to be a vector subspace of V is that it is closed under addition and multiplication.

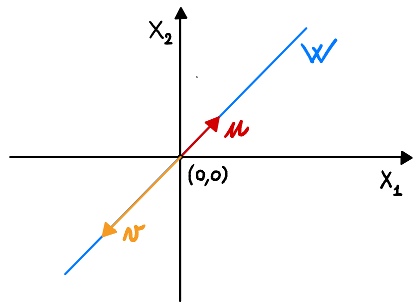
Example 1: A line passing through the origin of a Cartesian coordinate system is a vector subspace of $\mathbb{R}^2$. In fact, such line is defined by the set of points $(x_1, x_2) \in \mathbb{R}^2$ satisfying the equation $a_1 x_1 + a_2 x_2 = 0$ (for some $a_1, a_2 \in \mathbb{R}$). As we shall see hereafter, the set

$$W = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, a_1 x_1 + a_2 x_2 = 0\}, \tag{4}$$

which represents the line, can be equivalently written as (assuming $a_2 \neq 0$)

$$W = \left\{ u \in \mathbb{R}^2 \,:\, u = x(1, -a_1/a_2), \quad x \in \mathbb{R} \right\}. \tag{5}$$

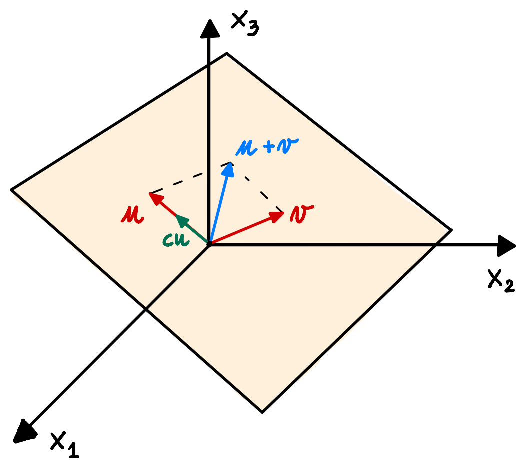
Clearly, W is a vector subspace of $\mathbb{R}^2$. In fact, 1) the zero of $\mathbb{R}^2$ is in W (the line passes through the origin); 2) a rescaling of a vector u on the line W is either zero or a vector that is still on the line; 3) the addition of two vectors u and v on the line is either zero or it is a vector that sits on the same line.



Example 2: A plane passing through the origin of a three-dimensional Cartesian coordinate system is a vector subspace of $\mathbb{R}^3$. Such plane can be defined as

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, a_1x_1 + a_2x_2 + a_3x_3 = 0 \qquad a_1, a_2, a_3 \in \mathbb{R}\}. \tag{6}$$

Clearly, W is a vector subspace of $\mathbb{R}^3$. In fact, 1) the zero of $\mathbb{R}^3$ is in W (the plane passes through the origin); 2) a rescaling of a vector u on the plane is either zero or a vector that is still on the plane; 3) the addition of two vectors on the plane is either zero or a vector on the plane.



Example 3: The space of continuously differentiable functions is a vector subspace of the space of continuous functions. In fact: 1) the addition between two differentiable functions $f(x)$ and $g(x)$ is a differentiable function $f(x) + g(x)$; 2) multiplication of a differentiable function $f(x)$ by a scalar c is a differentiable function $cf(x)$.

Example 4: The space 3×3 symmetric matrices is a vector subspace of $M_{3\times3}(\mathbb{R})$. In fact, if A and B are symmetric then: 1) $A + B$ is symmetric, 2) the zero matrix $0_{M_{3\times3}}$ is symmetric, and 3) cA is symmetric for all $c \in \mathbb{R}$.

Example 5: The space of polynomials of degree at most 3, i.e., $\mathbb{P}_3(\mathbb{R})$, is a vector subspace of the space of polynomials of degree at most 8, i.e., $\mathbb{P}_8(\mathbb{R})$.

Linear combination. Let V be a vector space over K. A linear combination of $v_1, \dots, v_n \in V$ is an expression of the form

$$x_1 v_1 + \cdots + x_n v_n. \tag{7}$$

We say that the set of vectors $v_1, \dots, v_n \in V$ *generates* V if for every $v \in V$ there exist n numbers $x_1, \dots, x_n \in K$ such that

$$v = x_1 v_1 + \cdots x_n v_n. \tag{8}$$

Example 1: The vectors

$$v_1 = (1,0), \qquad v_1 = (1,1), \tag{9}$$

generate $\mathbb{R}^2$.

Example 2: The matrices

$$v_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \tag{10}$$

generate the space of 2×2 matrices with real coefficients $M_{2\times 2}(\mathbb{R})$. Similarly, the matrices

$$v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{11}$$

generate the space of 2×2 *symmetric* matrices.

Example 3: The polynomials

$$p_1(x) = 1, \qquad p_2(x) = x \qquad p_3(x) = x^2 \tag{12}$$

generate the vector space of polynomials of degree at most 2.

Definition. Let V be a vector space over K. The space generated by $v_1, \dots, v_p \in V$ is called *span* of $v_1, \dots, v_p$ and denoted by $\mathrm{span}\{v_1, \dots, v_p\}$.

Theorem 1. Let V be a vector space over K. The span of an arbitrary number of vectors $v_1, ..., v_p \in V$ is a vector subspace of V.

Proof. Let $v_1, \dots, v_p$ be vectors in V. Consider the space generated by $v_1, \dots, v_p$, i.e.,

$$W = \mathrm{span}\{v_1, \dots, v_p\} = \{v \in V : v = x_1 v_1 + \cdots + x_p v_p, \quad x_i \in K\}. \tag{13}$$

and pick two elements in W

$$u = x_1 v_1 + \cdots + x_p v_p, \qquad v = y_1 v_1 + \cdots + y_p v_p. \tag{14}$$

Clearly, $0_V \in Q$, $(u+v) \in W$, and $cu \in W$ (for all $c \in K$).

□

By using the last theorem we immediately see why lines and planes are vector subspaces of $\mathbb{R}^3$. In fact, a line is a vector subspace generated by a nonzero vector $u \in \mathbb{R}^3$. Specifically, consider the line $(x_1, -3x_1, 2x_1)$ (for all $x_1 \in R$). This line is generated by the vector $u = (1, -3, 2)$. Similarly, the plane $x_1 + x_2 - 2x_3 = 0$ is generated, e.g., by the two vectors $v_1 = (1, 1, 1)$ and $v_2 = (2, 0, 1)$. In fact, any element on the plane can be expressed as a linear combination of v_1 and v_2.

Linear independence. Let V be a vector space over K, $v_1, \ldots, v_n \in V$. We say that n vectors $v_1, \ldots, v_n$ are linearly *independent* if

$$x_1 v_1 + \cdots + x_n v_n = 0_V \quad \Rightarrow \quad x_1, \ldots, x_n = 0 \tag{15}$$

Example 1: The following vectors of $\mathbb{R}^2$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{16}$$

are linearly independent. In fact,

$$x_1 v_1 + x_2 v_2 = 0_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \tag{17}$$

Example 2: The following two vectors of $\mathbb{R}^3$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{18}$$

are linearly independent. In fact,

$$x_1 v_1 + x_2 v_2 = 0_{\mathbb{R}^2} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

Let us compute the reduced row echelon form of the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 0 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]. \tag{20}$$

Hence, the system is consistent (see the last row), i.e., it has a solution. Moreover, the solution is unique and given by $x_1 = x_2 = 0$.

Example 3: The following 2×2 matrices

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \tag{21}$$

are *linearly dependent.* In fact,

$$x_1 A + x_2 B + x_3 C = 0_{M_{2\times 2}} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 + x_2 + 2x_3 & x_1 + x_2 + 2x_3 \\ 2x_1 + x_2 + 3x_3 & 3x_1 + x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

which yields the system

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \\ 3x_1 + x_2 + 4x_3 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \quad \text{free} \end{cases} \tag{23}$$

Therefore the condition $x_1 A + x_2 B + x_3 C = 0_{M_{2\times 2}}$ implies that

$$x_3 A + x_3 B = x_3 C \qquad \forall x_3 \in \mathbb{R} \tag{24}$$

and therefore the matrices A, B and C are linearly dependent.

Basis of a vector space. Let V be a vector space over K. A basis of V is a set of linearly independent vectors in V that generate V.

Example: The vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{25}$$

are a basis for the vector space $\mathbb{R}^2$. In fact, they are linearly independent and they generate $\mathbb{R}^2$. To show that they generate $\mathbb{R}^2$ we need to show that every vector $u \in \mathbb{R}^2$ can be represented as a linear combination of v_1 and v_2. In other words, given $u \in \mathbb{R}^2$ we need to show that there exist x_1 and x_2 such that

$$x_1 v_1 + x_2 v_2 = u. \tag{26}$$

This is equivalent to show that the following linear system of equations has a unique solution

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{27}$$

which is obvious since the matrix of coefficients is invertible.

Definition (Coordinates relative to a basis)**.** Let V be a vector space over K, $v_1, \ldots, v_n \in V$ a basis for V, and $v \in V$. The numbers $x_1, \ldots, x_n$ such that $v = x_1 v_1 + \cdots + x_n v_n$ are called *coordinates* of v relative $v_1, \ldots, v_n$.

Theorem 2. The coordinates of an arbitrary vector v in a vector space V are uniquely determined by the basis.

Proof. Let $v_1, \dots, v_n$ be a basis for V. Suppose that for some $v \in V$ there are two set of coordinates $\{x_i\}$ and $\{y_i\}$ such that

$$v = x_1 v_1 + \cdots + x_n v_n = y_1 v_1 + \cdots + y_n v_n \quad \Rightarrow \quad (x_1 - y_1)v_1 + \cdots + (x_n - y_n)v_n = 0_V. \tag{28}$$

This implies that $x_i = y_i$ since the vectors $v_1, \dots, v_n$ are linearly independent.

□

Example: The coordinates of $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ relative to $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} -10 \\ 9 \end{bmatrix}$ can be computed by solving the linear system of equations

$$x_1 v_1 + x_2 v_2 = v \quad \Rightarrow \quad \begin{bmatrix} 1 & -10 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{29}$$

Example: Find the coordinates of $p(x) = x^3 + x + 1$ relative to the following basis of $\mathbb{P}_3(\mathbb{R})$

$$p_0(x) = 5, \qquad p_1(x) = x, \qquad p_2(x) = x^2 + 1, \qquad p_3(x) = x^3 - x^2. \tag{30}$$

Let $y_0, \dots, y_3$ be the coordinates of $p(x)$ relative to $\{p_0(x), \dots, p_3(x)\}$. We have,

$$y_0 p_0(x) + \cdots + y_3 p_3(x) = x^3 + x + 1. \tag{31}$$

Developing the products we find

$$y_3 x^3 + (y_2 - y_3)x^2 + y_1 x + (5y_0 + y_2) = x^3 + x + 1, \tag{32}$$

Which yields the linear system

$$\begin{cases} y_3 = 1 \\ y_2 - y_3 = 0 \\ y_1 = -1 \\ 5y_0 + y_2 = 1 \end{cases} \qquad \Rightarrow \qquad \begin{cases} y_3 = 1 \\ y_2 = 1 \\ y_1 = -1 \\ y_0 = 0 \end{cases} \tag{33}$$

Example: The coordinates of the symmetric matrix

$$v = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}, \tag{34}$$

relative to the basis v_1, v_2 and v_3 defined in Eqs. (11) are $x_1 = -2$, $x_2 = 4$ and $x_3 = 3$.

Dimension of a vector space. The dimension of a vector space V is the number of linearly independent vectors required to generate V, i.e., the number of elements in any basis of V. We denote the dimension of V as $\dim(V)$. We have, for example,

- $\dim\left(M_{2\times 2}(\mathbb{R})\right) = 4$,
- $\dim\left(\mathbb{R}^3\right) = 3$,
- $\dim\left(\mathbb{P}_4(\mathbb{R})\right) = 5$,
- $\dim\left(C^{(1)}(\mathbb{R})\right) = \infty$

It is easy to show that if $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ are $m > n$ vectors of V then $w_1, \ldots, w_m$ are necessarily linear dependent. This means that number of vectors in every basis of V is that minimum one that is needed to generate V. To show this, we let us write each vector w_i in terms of the basis

$$\begin{cases} w_1 = x_{11}v_1 + \cdots + x_{1n}v_n \\ \vdots \\ w_m = x_{m1}v_1 + \cdots + x_{mn}v_n \end{cases} \tag{35}$$

Now, suppose that $w_1, \ldots, w_m$ are linearly independent, i.e.,

$$0_V = y_1 w_1 + \cdots + y_m w_m \quad \Rightarrow \quad y_1, \ldots, y_m = 0. \tag{36}$$

By substituting (35) into (36) we obtain,

$$0_V = y_1 w_1 + \cdots y_m w_m = (y_1 x_{11} + \cdots + y_m x_{m1})v_1 + \cdots + (y_1 x_{1n} + \cdots + y_m x_{mn})v_n, \tag{37}$$

which implies that

$$\begin{cases} y_1 x_{11} + \cdots + y_m x_{m1} = 0 \\ \vdots \\ y_1 x_{1n} + \cdots + y_m x_{mn} = 0 \end{cases} \tag{38}$$

This is a homogeneous linear system of $n < m$ equation in m unknowns $(y_1, \ldots, y_m)$. which always admits a nontrivial (i.e., nonzero) solution. Hence, $y_1, ..., y_m$ cannot be all zero, and therefore $w_1, ..., w_m$ are necessarily linearly dependent.

We conclude this section by emphasizing that a set of p linearly independent vectors in a vector space V of dimension $n > p$ can be always complemented with additional linearly independent vectors to become a basis of V.

The rank of a matrix. Consider the following $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \tag{39}$$

The columns of A generate a vector space called *column space of* A. Similarly, the rows of A generate a vector space called *row space of* A

$$\text{Column space of } A: \quad \text{span} = \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}. \tag{40}$$

$$\text{Row space of } A: \quad \text{span} = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix}, \begin{bmatrix} a_{21} & \cdots & a_{2n} \end{bmatrix}, \dots, \begin{bmatrix} a_{m1} & \cdots & a_{mn} \end{bmatrix} \right\}. \tag{41}$$

Note that the column space of A is a vector subspace of $\mathbb{R}^m$, while the row space of A is a vector subspace of $\mathbb{R}^n$.

The dimension of the column space is called *column rank*, while the dimension of the row space is called *row rank*. Both ranks can be computed by reducing the matrix to an echelon form using elementary row or column operations, i.e.,

1. Adding a scalar multiple of one row (column) to another row (column);
2. Interchange rows (columns),
3. Multiplying one row (column) by a non-zero number.

Theorem 3. Elementary row or column operations do not change the row rank nor the column rank of a matrix[2].

This statement follows immediately by noting that linear taking linear combinations of a fixed number of vectors does not change the dimension of the span of such vectors. Moreover, taking permutations of the entries of a set of vectors in the same way for all vectors does not alter linear independence.

By performing both rows and column operations it is possible to transform any $m \times n$ matrix into the following canonical form (block matrix)

$$A = \begin{bmatrix} I_r & 0_{M_{r\times(n-r))}} \\ 0_{M_{(m-r)\times r}} & 0_{M_{(m-r)\times(n-r)}} \end{bmatrix}, \tag{42}$$

where I_r is a $r \times r$ identity matrix, and all other matrices are zero matrices.

This means that the dimension of the row space of a matrix is always the same as the dimension of the column space. Phrasing this differently:

Theorem 4. The row rank of a matrix is always the same as the column rank.

Hence, we can omit "row" or "column" and just speak of the *rank of a matrix*. Clearly, for an $m \times n$ matrix the rank r is always smaller or equal than the minimum between the number of rows m and the number of columns n, i.e.,

$$r \leq \min\{m, n\}. \tag{43}$$

Example 1: By using elementary row and column operations reduce the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 & -1 \\ 0 & 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 4 & -1 \end{bmatrix} \tag{44}$$

[2]Note that elementary column operations can change the solution to a linear system of equations. In fact, if we perform Gauss elimination along a row we are essentially eliminating the coefficient multiplying, say, x_k using the coefficient of the variable x_j. Clearly, this changes the solution of the linear system.

to the canonical form (42).

$$\begin{bmatrix}1 & 1 & 1 & 2 & -1\\0 & 2 & 1 & 0 & 1\\2 & 0 & 1 & 4 & -1\end{bmatrix} \xrightarrow{R_3:R_3-2R_1} \begin{bmatrix}1 & 1 & 1 & 2 & -1\\0 & 2 & 1 & 0 & 1\\0 & -2 & -1 & 0 & 1\end{bmatrix} \xrightarrow[C_4:C_4-2C_1]{C_2:C_2-C_1} \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 2 & 1 & 0 & 1\\0 & -2 & -1 & 0 & 1\end{bmatrix} \tag{45}$$

$$\xrightarrow{R_3:R_3+R_2} \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 2 & 1 & 0 & 1\\0 & 0 & 0 & 0 & 2\end{bmatrix} \xrightarrow[C_2:C_2-2C_1]{C_4:C_4-C_2} \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 2\end{bmatrix} \xrightarrow[C_2\leftrightarrow C_3]{C_3\leftrightarrow C_5/2} \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\end{bmatrix} \tag{46}$$

Hence, the rank of the matrix (44) is $r = 3$.

Example 2: Find the rank of the matrix

$$A = \begin{bmatrix}1 & 3 & 4\\1 & 1 & -1\\2 & 6 & 0\end{bmatrix}. \tag{47}$$

A is row equivalent to the following matrix[3]

$$A = \begin{bmatrix}1 & 3 & 4\\0 & -2 & -5\\0 & 0 & -8\end{bmatrix}. \tag{48}$$

Clearly, the columns of this matrix are linearly independent and therefore the rank is 3.

Example: The rank of the following matrices is equal to 2

$$A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \quad B = \begin{bmatrix}1 & 2 & 1\\0 & 0 & 1\\-1 & -2 & 0\end{bmatrix}, \quad C = \begin{bmatrix}1 & 1 & 1 & 2 & -1\\0 & 2 & 1 & 0 & -1\\2 & 0 & 1 & 4 & -1\end{bmatrix}, \quad D = \begin{bmatrix}1 & 1\\0 & 2\\2 & 0\\1 & 1\end{bmatrix}. \tag{49}$$

[3]Recall that elementary row or column operations do not change the rank of a matrix.