Lecture 9: Determinants

Let $A \in M_{n \times n}$ be a square matrix with real or complex entries

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad (1)$$

The determinant of $A$ is the real or complex number

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A^{ij}) \quad \text{(for every fixed } i), \quad (2)$$

where $a_{ij}$ are the entries of $A$, and $A^{ij}$ is a matrix obtained from $A$ by crossing out the $i$-th row and the $j$-th column$^1$.

The expression (2) is called Laplace expansion of the determinant along the $i$-th row. As we will see hereafter $\det(A) = \det(A^T)$ and therefore there exists an equivalent Laplace expansion along the $j$-th column, which is

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A^{ij}) \quad \text{(for every fixed } j). \quad (4)$$

Remark: The fact that we can arbitrarily choose the row or the column along which develop the determinant and always obtain the same result suggests that the determinant is a rather special function. From a technical viewpoint it can be shown that (2) is the a unique alternating multilinear function$^2$

$$\Lambda(\cdot) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}$$

n times

$$(a_1, \ldots, a_n) \rightarrow \Lambda(a_1, \ldots, a_n) \quad (6)$$

satisfying

$$\Lambda(e_1, \ldots, e_n) = 1, \quad (7)$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. In other words, we have

$$\det(A) = \Lambda(a_1, \ldots, a_n), \quad (8)$$

where $a_j$ is the $j$-th column of $A$.

$^1$The number

$$c_{ij} = (-1)^{i+j} \det(A^{ij}) \quad (3)$$

is often called co-factor of $a_{ij}$ in the determinant expansion.

$^2$An alternating multilinear function is a function $\Lambda(a_1, \ldots, a_n)$ that is linear in each argument $a_j$, e.g.,

$$\Lambda(a_1, a_2 + b_2, a_3) = \Lambda(a_1, a_2, a_3) + \Lambda(a_1, b_2, a_3),$$

and changes sign if we interchange $a_j$ with $a_i$. For instance,

$$\Lambda(a_1, a_2, a_3) = -\Lambda(a_2, a_1, a_3) = \Lambda(a_3, a_1, a_2). \quad (5)$$
If $A = a$ is a number then we set $\det(A) = a$. Note that the determinant of a matrix is a \textit{nonlinear} function of the matrix entries which is defined recursively in terms of determinants of the matrices $A^i_j$, which have smaller dimension.

\textbf{Examples:} Let us provide a few examples of calculation of determinants

1. $A = a$ (real number). In this case we have $\det(A) = a$.

2. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We develop the determinant along the first row, i.e., set $i = 1$ in (2).

This yields
\[ \det(A) = (-1)^{1+1} a_{11} \det(A^{11}) + (-1)^{1+2} a_{12} \det(A^{12}), \tag{9} \]
where
\[ A^{11} = a_{22}, \quad A^{12} = a_{21}. \tag{10} \]

Therefore we obtain
\[ \det(A) = a_{11}a_{22} - a_{12}a_{21}. \tag{11} \]

Note that we obtain exactly the same formula if we develop the determinant along the second row, the first column or the second column.

3. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. We develop the determinant along the first row, i.e., set $i = 1$ in (2).

This yields
\[ \det(A) = (-1)^{1+1} a_{11} \det(A^{11}) + (-1)^{1+2} a_{12} \det(A^{12}) + (-1)^{1+3} a_{13} \det(A^{13}) \tag{12} \]
where
\[ A^{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad A^{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad A^{11} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}. \tag{13} \]

Computing the determinants of $A^{11}$, $A^{12}$ and $A^{13}$ yields the formula
\[ \det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}). \tag{14} \]

Note that we obtain exactly the same formula if we develop the determinant along any other row or column.

Since we can equivalently expand the determinant along arbitrary rows or columns of $A$ it is convenient to choose the row or the column with the largest number of zeros. This minimizes the number of calculations when computing the determinant using (2) or (4). For example, it is clear that it is convenient to compute the determinant of the following matrix along the second column:
\[ \det \left( \begin{bmatrix} 3 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 5 \end{bmatrix} \right) = -1 \det \left( \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix} \right) = -1(-5 - 2) = 7. \tag{15} \]
Properties of the determinant. The determinant of any \( n \times n \) matrix \( A \) satisfies the following important properties:

1. \( \det(A) = \det(A^T) \)
2. \( \det(A) \) is a linear function of the columns (or the rows) of the matrix \( A \). In other words, if we denote by \( a_i \) the \( i \)-th column of \( A \) and \( B \) a column vector of the same length of \( a_i \) then
   
   (a) \( \det \left( \begin{bmatrix} a_1 & \cdots & (a_i + b) & \cdots & a_n \end{bmatrix} \right) = \det \left( \begin{bmatrix} a_1 & \cdots & a_i & \cdots & a_n \end{bmatrix} \right) + \det \left( \begin{bmatrix} a_1 & \cdots & b & \cdots & a_n \end{bmatrix} \right), \)
   
   (b) \( \det \left( \begin{bmatrix} a_1 & \cdots & ca_i & \cdots & a_n \end{bmatrix} \right) = c \det \left( \begin{bmatrix} a_1 & \cdots & a_i & \cdots & a_n \end{bmatrix} \right), \)

   where \( a_i \) is the \( i \)-th column of \( A \).
3. If the columns or the rows of \( A \) are linearly dependent then \( \det(A) = 0 \). If the columns or the rows or \( A \) are linearly independent (i.e., \( A \) is full rank) then \( \det(A) \neq 0 \).
4. If a multiple of one row (or one column) is added or subtracted to another row (or column) then the determinant does not change (this follows from property 2 by setting with \( B = cA_j \), and property 3).
5. If two rows (or two columns) are interchanged then the determinant changes sign.

These properties can be easily verified for \( 2 \times 2 \) and \( 3 \times 3 \) matrices. The proof of these properties for general \( n \times n \) matrices can be found in the book.

Note that from property 2(b) it follows that for any number \( c \) and any \( n \times n \) matrix \( A \):

\[
\det(cA) = c^n \det(A). \tag{16}
\]

In fact, the matrix \( cA \) has all columns \( (n \text{ in total}) \) multiplied by \( c \).

Example: Let us show properties 1. to 5. for the simple matrix

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \det(A) = 1. \tag{17}
\]

We have:

1. \( \det \left( A^T \right) = \det \left( \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \right) = 1 = \det(A) \).
2. (a) \( \det \left( \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \right) = \det \left( \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \right) + \det \left( \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \right) = 3 - 2 = 1. \)
   
   (b) Multiply the second column of \( A \) by 3. This yields \( \det \left( \begin{bmatrix} 1 & 3 \\ 2 & 9 \end{bmatrix} \right) = 3 = 3 \det(A). \) \( \underbrace{= 1}_{1} \)
3. \( A \) has rank 2 and therefore its columns are linearly independent. However, if we consider the rank 1 matrix \( \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \) then

\[
\det \left( \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \right) = 3 - 3 = 0. \tag{18}
\]
4. Multiply the first row of $A$ by 2 and add it to the second row to obtain

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}\right) = 7 - 6 = 1 = \det(A).$$

(19)

5. Interchange the first and the second row of $A$ to obtain

$$\det\left(\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\right) = 2 - 3 = -1 = -\det(A).$$

(20)

From Property 3. at page 3 it follows that:

**Theorem 1.** Let $A$ be a $n \times n$ matrix, $b$ a $n \times 1$ column vector. Then

1. $A$ is invertible $\iff \det(A) \neq 0$.
2. $A$ has linearly independent rows/columns $\iff \det(A) \neq 0$.
3. The linear system of equations $Ax = b$ has a unique solution $\iff \det(A) \neq 0$.

**Determinant of matrix products and matrix inverses.** By using the definition of determinant (2) it can be shown that for every $A, B \in M_{n \times n}$ we have

$$\det(AB) = \det(A) \det(B).$$

(21)

Clearly, this implies that

$$\det(AB) = \det(BA) \quad \text{and} \quad \det(A^p) = \det(A)^p \quad \text{for all } p \in \mathbb{N}. \quad (22)$$

By using these identities it is straightforward to show, e.g.,

$$\det(AB^TACAB) = \det(A)^3 \det(B)^2 \det(C),$$

(23)

where $A$, $B$, and $C$ are three $n \times n$ matrices. Moreover, if $A$ is an invertible matrix then

$$1 = \det(AA^{-1}) = \det(A) \det(A^{-1}) \quad \Rightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)},$$

(24)

i.e. the determinant of the inverse matrix is the inverse of the determinant.

**Computing the determinant of a matrix efficiently.** How do we actually compute the determinant of a matrix? We have seen that one possibility is to use the definition (2), i.e., the Laplace rule. However, this not really computationally efficient if the dimension of the matrix is even moderately high, e.g., larger than 10 or 20. In fact, it can be shown that the number of operations to compute (2) is exactly

$$p = \lfloor n!e \rfloor - 2. \quad (25)$$

In this formula, $e = 2.7183...$ is the Napier number and the symbol $\lfloor n!e \rfloor$ denotes the nearest integer number smaller or equal than $n!e$, where $n!$ is the factorial of $n$. For instance, if $n = 2$ we have

$$p = \lfloor 2!e \rfloor - 2 = \lfloor 5.4366 \rfloor - 2 = 5 - 2 = 3.$$
In fact, as we see from equation (11), to compute the determinant we need two multiplications and one subtraction. Similarly, for $3 \times 3$ matrices ($n = 3$) we need


operations. In fact, as we see from equation (12), to compute the determinant we need 9 multiplications and 5 subtractions. The number of operations increases exponentially fast as we increase the dimension of the matrix. For example, for a $20 \times 20$ matrix the Laplace rule (2) requires

$$p = \lfloor 20!e \rfloor - 2 \approx 6.61 \times 10^{18}$$

operations. The 2022 Apple M2 Max processor is capable of 13.6 Teraflops in single precision (32 bits), i.e., $13.6 \times 10^{12}$ single-precision floating point operations per second. Hence, to compute the determinant of a $20 \times 20$ matrix by using the Laplace rule on the latest MacBook Pro with M2 Max processor/GPU we need to let our laptop run for approximately

$$\frac{6.61 \times 10^{18}}{13.6 \times 10^{12} \text{ flops}} = 4.86 \times 10^5 \text{ seconds} \approx 5.62 \text{ days}$$

(26) to complete the calculation. Repeating a similar calculation for a $22 \times 22$ would require

$$\frac{30.554 \times 10^{20}}{13.6 \times 10^{12} \text{ flops}} = 2.2632 \times 10^8 \text{ seconds} \approx 7.18 \text{ years}.$$  

(27)

Fortunately, there is a more efficient algorithm to compute the determinant of a matrix. In fact, by using elementary row operations we known that we can reduce the matrix $A$ to the following matrix in row-echelon form

$$U = \begin{bmatrix}
u_{11} & u_{12} & \cdots & u_{1n} \\
0 & u_{12} & \cdots & u_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & u_{nn}
\end{bmatrix}$$

(28)

The matrix $U$ has the same determinant of $A$, up to a sign (i.e., + or −) determined by how many times we interchange rows in the Gauss elimination with pivoting-by-row process. If we denote by $s$ the number of row permutations we take in the Gauss elimination process we have

$$\det(A) = (-1)^s \prod_{k=1}^{n} u_{kk}.$$ 

(29)

In fact, the determinant of an upper-triangular (or a lower-triangular) matrix is simply the product of the diagonal elements. The total number of operations to transform an $n \times n$ matrix $A$ into the upper triangular form $U$ is

$$\frac{2}{3}n^3 - \frac{n^2}{2} - \frac{n}{6}.$$ 

(30)

The number of products in (29) is $n$, while taking the exponential is one operation. Hence, the total number of operations to compute the determinant with Gauss elimination is

$$\frac{2}{3}n^3 - \frac{n^2}{2} - \frac{n}{6} + n + 1 = \frac{2}{3}n^3 - \frac{1}{2}n^2 + \frac{5}{6}n + 1$$

(31)
For a $22 \times 22$ matrix we get 6876 operations. If we use a 2022 Apple M2 Max processor, this requires
\[
\frac{6876 \text{ operations}}{13.6 \times 10^{12} \text{ flops}} = 5.06 \times 10^{-10} \text{ seconds.} \tag{32}
\]

**Example:** Consider the matrix
\[
A = \begin{bmatrix}
1 & -1 & 3 \\
1 & -2 & 1 \\
-1 & 1 & 3
\end{bmatrix} \quad \rightarrow \quad \text{Gauss elimination} \quad \rightarrow \quad U = \begin{bmatrix}
1 & -1 & 3 \\
0 & -1 & -2 \\
0 & 0 & 6
\end{bmatrix} \tag{33}
\]
Since we did not perform any permutation we have $s = 0$ in (29) and therefore
\[
\det(A) = \det(U) = -6.
\]

**Cramer’s rule.** It is possible to express the solution to a linear system of equations in terms of determinants. Specifically, let
\[
Ax = b
\]
be a system of $n$ linear equations in $n$ unknowns. Suppose that the system has a unique solution (i.e., $\det(A) \neq 0$). Then
\[
x_i = \frac{1}{\det(A)} \det([a_1 \cdots b \cdots a_n]),
\]
where $[a_1 \cdots b \cdots a_n]$ is a matrix obtained by replacing the $i$-th column of $A$ (denoted by $a_i$) with the column vector $b$.

**Example:** Compute the solution to the following system of equations using Cramer’s rule:
\[
\begin{bmatrix}
1 & 2 \\
1 & 3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 \\
2
\end{bmatrix}. \tag{36}
\]
We have $\det(A) = 1$, and therefore
\[
x_1 = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\right) = -1, \quad x_2 = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) = 1. \tag{37}
\]

**An explicit formula for $A^{-1}$.** Let $A$ be an invertible matrix. By definition, the inverse of $A$ is a square matrix (denoted as $A^{-1}$) with the following properties
\[
AA^{-1} = I_n \quad A^{-1}A = I_n, \tag{38}
\]
where $I_n$ is the $n \times n$ identity matrix. Let $h_i$ be the columns of the matrix $A^{-1}$, i.e.,
\[
A^{-1} = [h_1 \ h_2 \ \cdots \ h_n] \quad h_i \in M_{n \times 1} \quad i = 1, \ldots, n. \tag{39}
\]
By definition of matrix-vector product we have
\[
AA^{-1} = [Ah_1 \ Ah_2 \ \cdots \ Ah_n]. \tag{40}
\]
At this point, define the following column vectors $e_i \in M_{n \times 1}$ ($i = 1, \ldots, n$)

\[
e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.
\] (41)

Note that $e_i$ is the $i$-th column of the identity matrix $I_n$. With this notation we can write the matrix equation $AA^{-1} = I_n$ as

\[
[Ah_1 \ Ah_2 \ \cdots \ Ah_n] = [e_1 \ e_2 \ \cdots \ e_n].
\] (42)

Hence, the $n$ columns of the inverse matrix $A^{-1}$, i.e., $h_1, \ldots, h_n$ are solutions to $n$ linear systems

\[
Ah_1 = e_1, \quad Ah_2 = e_2, \quad \ldots, \quad Ah_n = e_n.
\] (43)

By using Cramer’s rule we obtain that the $i$-th component of the column vector $h_j$ is

\[
h_{ji} = \frac{1}{\det(A)} \det([a_1 \cdots e_j \cdots a_n])
\] (44)

where $[a_1 \cdots e_j \cdots a_n]$ is a matrix in which we replaced the $i$-th column $a_i$ with $e_j$. By using the Laplace rule along the $i$-th column of $[a_1 \cdots e_j \cdots a_n]$ we obtain

\[
\det([a_1 \cdots e_j \cdots a_n]) = (-1)^{i+j} \det(A^{ji}) = C^{ji} \quad (j, i)$-cofactor.
\] (45)

This yields the following expression

\[
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C^{11} & \cdots & C^{1n} \\ \vdots & \ddots & \vdots \\ C^{n1} & \cdots & C^{nn} \end{bmatrix}^T.
\] (46)

**Example:** Compute the inverse of the following matrix

\[
A = \begin{bmatrix} -2 & 4 \\ 4 & 3 \end{bmatrix}.
\] (47)

We have $\det(A) = -22$, and

\[
C^{11} = \det([e_1 \ a_2]) = \det\left( \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \right) = 3,
\]

\[
C^{12} = \det([a_1 \ e_1]) = \det\left( \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \right) = -4,
\]

\[
C^{21} = \det([e_2 \ a_2]) = \det\left( \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} \right) = -4,
\]

\[
C^{22} = \det([a_1 \ e_2]) = \det\left( \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} \right) = -2.
\]

Therefore,

\[
A^{-1} = \frac{1}{-22} \begin{bmatrix} -3 & 4 \\ 4 & 2 \end{bmatrix}.
\] (48)
**Volumes of parallelograms.** The determinant of a matrix represents the volume enclosed by the vectors defined by the columns (or the rows) or the matrix.

\[ A = |\det([v_1, v_2])| \quad \text{and} \quad V = |\det([v_1, v_2, v_3])| \]

At this point we notice that there are quite a lot of properties of \( A \) and \( V \) following from the properties of determinant. For example, if we add a scalar multiple of \( v_1 \) to \( v_2 \), then the area of the parallelogram defined by the two vectors does not change. This follows from the fact that the determinant is a linear function of the columns, and that the determinant of a matrix with linearly dependent columns is equal to zero. For example,

\[ A = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}. \quad (49) \]

Of course, the green and blue areas are the same. Other properties of the area of a parallelogram can be derived from properties of the determinant. Next, consider an invertible transformation \( F: \mathbb{R}^n \to \mathbb{R}^n \), represented by \( n \times n \) invertible matrix \( L \). We know that if \( \{v_1, \ldots, v_n\} \) is a basis of \( \mathbb{R}^n \) then

\[ [u_1 \ldots u_n] = [Lv_1 \ldots Lv_n] = L[v_1 \ldots v_n] \quad (50) \]

is also a basis of \( \mathbb{R}^n \). The volume of the parallelograms enclosed by \( \{v_1, \ldots, v_n\} \) and \( \{u_1, \ldots, u_n\} \) are

\[ V_0 = |\det ([v_1 \ldots v_n])|, \quad V_1 = |\det ([u_1 \ldots u_n])|. \quad (51) \]

By applying the determinant operator to equation (50), and using the fact that the determinant of the matrix product is the product of the matrix determinants we see that

\[ V_1 = |\det(L)| V_0. \quad (52) \]

This formula is very important in a variety of fields ranging from multi-dimensional integration theory to continuum mechanics.