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A topological delay embedding theorem for infinite-dimensional dynamical systems

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Abstract

A time delay reconstruction theorem inspired by that of Takens (1981 *Springer Lecture Notes in Mathematics* vol 898, pp 366–81) is shown to hold for finite-dimensional subsets of infinite-dimensional spaces, thereby generalizing previous results which were valid only for subsets of finite-dimensional spaces.

Let $\mathcal{A}$ be a subset of a Hilbert space H with upper box-counting dimension $d(\mathcal{A}) = d$ and 'thickness exponent' τ, which is invariant under a Lipschitz map Φ. Take an integer $k > (2 + \tau)d$, and suppose that $\mathcal{A}_p$, the set of all p-periodic points of Φ, satisfies $d(\mathcal{A}_p) < p/(2+\tau)$ for all $p = 1, \ldots, k$. Then a prevalent set of Lipschitz observation functions $h : H \to \mathbb{R}$ make the k-fold observation map

$$u \mapsto [h(u), h(\Phi(u)), h(\Phi^{k-1}(u))],$$

one-to-one between $\mathcal{A}$ and its image. The same result is true if $\mathcal{A}$ is a subset of a Banach space provided that $k > 2(1 + \tau)d$ and $d(\mathcal{A}_p) < p/(2 + 2\tau)$.

The result follows from a version of the Takens theorem for Hölder continuous maps adapted from Sauer *et al* (1991 *J. Stat. Phys.* **65** 529–47), and makes use of an embedding theorem for finite-dimensional sets due to Hunt and Kaloshin (1999 *Nonlinearity* **12** 1263–75).

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1. Introduction

It is natural to ask whether the underlying dynamics of a physical system can be reconstructed from an experimental time series. Setting this in a sufficiently abstract framework to allow for a mathematical treatment, the question can be rephrased as follows. Suppose that the experimental set-up can be modelled by a dynamical system that evolves in some state space E. Given an observation function $h : E \to \mathbb{R}$, can a state in E be distinguished by repeated measurements of h along its future trajectory? (Such observability problems are common in control theory, see Sontag (2002) for example.)

This question was first answered positively by Aeyels (1981), but it is the following result due to Takens, which gives more information and guarantees that an 'accurate reconstruction' is possible given a sufficient number of observations at equally spaced times, that is now better known.

Theorem 1.1 (Takens 1981). *Let M be a compact manifold of dimension d. For pairs (φ, h), where $\varphi : M \to M$ is a smooth (at least C^2) diffeomorphism and $h : M \to \mathbb{R}$ a smooth function, it is a generic property that the $(2d+1)$-fold observation map $H_k[\varphi, h] : M \to \mathbb{R}^{2d+1}$ defined by*

$$x \mapsto (h(x), h(\varphi(x)), \ldots, h(\varphi^{2d}(x)))$$

is an immersion (i.e. H_k is one-to-one between M and its image with both H_k and H_k^{-1} differentiable).

The theorem can be applied to time series by taking φ to be the time T map of the underlying (continuous time) dynamical system, i.e. $\varphi^j(x_0) = x(jT)$, where $x(\cdot)$ is the trajectory starting at x_0. The reconstruction that is then provided is 'accurate' in two ways. The first is topological: the map H_k is one-to-one between M and its image in $\mathbb{R}^{2d+1}$, so that the time delay coordinates

$$[h(x(0)), h(x(T)), \ldots, h(x(2dT))]$$

can be guaranteed to distinguish between points on M (the result of Aeyels proves only this one-to-one property).

The second is dynamical: the time T map on M is equivalent to a shift on the time series in 'delay coordinate space',

$$h(x(0)), \quad \underbrace{h(x(T)), \quad \ldots, \quad h(x(2dT)), \quad h(x((2d+1)T))}_{H_k(x(T))}, \quad h(x((2d+2)T)) \quad \ldots,$$

so we can hope to use these induced dynamics to obtain properties of the time T map on M. Since H_k is a diffeomorphism (its inverse is differentiable) this reconstruction preserves the dimension of any invariant set and the Lyapunov exponents of the flow.

Although the conclusions of this theorem are strong, so are its assumptions, which are hard to verify in general and may in fact fail in a number of practical applications. The requirement that the dynamics take place on a compact finite-dimensional manifold is very restrictive, and *a priori* excludes the application of the result to the infinite-dimensional dynamical systems arising from partial differential equations, and in particular fluid turbulence. This means that theorem 1.1 provides no rigorous justification for the use of time-delay reconstruction for data from many experimental situations.

It is therefore of some importance to try to generalize the Takens theorem to such infinite-dimensional systems. This paper presents a generalization of the one-to-one part of the theorem which can be applied to infinite-dimensional systems that have finite-dimensional attractors. Note, however, that the resulting dynamical reconstruction may distort the Lyapunov exponents and the dimension of invariant sets, since the observation mapping is not necessarily a diffeomorphism. (For related problems in a different setting see Robinson (1999).)

The argument—which is surprisingly simple given that this problem has been open for over twenty years[1]—relies heavily on the work of Sauer *et al* (1993), who proved a very similar result for finite-dimensional attractors of *finite-dimensional* dynamical systems, and the paper of Hunt and Kaloshin (1999) concerning the embedding of finite-dimensional sets into finite-dimensional spaces.

[1] Two recent papers on more abstract embeddings (Hunt and Kaloshin 1999, Mansfield *et al* 1999), highlight this as an important problem.

To be more mathematically precise, suppose that the underlying physical model generates a dynamical system on an infinite-dimensional Hilbert space H, with the solution at time t through the initial condition u_0 given by

$$u(t; u_0) = S(t)u_0.$$

The solution operator $S(\cdot) : H \to H$ forms a semigroup satisfying the properties

$$S(0) = \mathrm{id}, \qquad S(t)S(s) = S(t+s) \qquad \text{and} \qquad S(t)u_0 \quad \text{continuous in } t \text{ and } u_0.$$

Such a dynamical system is generated by many interesting partial differential equations, including the two-dimensional Navier–Stokes equations (see Temam (1988) or Robinson (2001) for details).

For many dissipative equations, it is possible to show that this dynamical system has a *global attractor* $\mathcal{A}$: a compact, positively invariant set which attracts (as $t \to \infty$) the orbits of all bounded sets. The 'asymptotic behaviour' of the system can then be regarded as the dynamics of $S(t)$ restricted to $\mathcal{A}$ (cf Hale (1988), Robinson (2001), Temam (1988)). In many cases (see Temam (1988) for numerous examples), these attractors can be shown to be finite-dimensional subsets of the ambient infinite-dimensional phase space, and this is the key to the treatment here.

As in the statement of theorem 1.1, it is convenient to work with iterated maps rather than systems evolving continuously in time. Although the main result of this paper applies to any Lipschitz map Φ, the application to time series makes the choice $\Phi = S(T)$ for some $T > 0$ very natural. A simplified statement of theorem 5.1 is the following.

Theorem 1.2. *Let $\mathcal{A}$ be a compact subset of a Hilbert space H with upper box-counting dimension $d(\mathcal{A}) = d$, which has thickness exponent zero, and is an invariant set for a Lipschitz map $\Phi : H \to H$. Choose an integer $k > 2d$, and suppose further that the set $\mathcal{A}_p$ of p-periodic points of Φ satisfies $d(\mathcal{A}_p) < p/2$. Then a prevalent set of Lipschitz maps $f : H \to \mathbb{R}$ make the k-fold observation map $D_k[f, \Phi] : H \to \mathbb{R}^k$ defined by*

$$D_k[f, \Phi](u) = (f(u), f(\Phi(u)), \ldots, f(\Phi^{k-1}(u))), \tag{1}$$

one-to-one on $\mathcal{A}$.

(The thickness exponent is defined in section 3; prevalence, a version of 'almost every' applicable in infinite-dimensional spaces, is defined in section 2; and the expression 'ϕ is one-to-one on X' will be used throughout this paper to mean that ϕ is one-to-one between X and its image.)

Broadly speaking, the theorem says that the dynamics on a finite-dimensional attractor can be reconstructed using a sufficient number of equally spaced observations, provided that there are not 'too many' periodic points of Φ. The key idea of the proof is to use an abstract embedding theorem due to Hunt and Kaloshin to reproduce the dynamics on $\mathcal{A}$ within a finite-dimensional space. Although the resulting dynamical system is not very smooth, a modified version of the result of Sauer *et al* can then be used to obtain the time delay embedding for the original system.

2. Prevalence

In line with the treatment in Sauer *et al* (1993) and in Hunt and Kaloshin (1999), the theorem here is expressed in terms of 'prevalence'. This concept, which generalizes the notion of 'almost every' from finite to infinite-dimensional spaces, was introduced by Hunt *et al* (1992); see their paper for a detailed discussion.

Definition 2.1. *A Borel subset S of a normed linear space V is* prevalent *if there is a finite-dimensional subspace E of V ('the probe space') such that for each $v \in V$, $v + e$ belongs to S for (Lebesgue) almost every $e \in E$.*

Note that if V is finite-dimensional then this corresponds (via the Fubini theorem) to S being a set whose complement has zero measure; and that if S is prevalent then S is dense in V.

3. Embedding finite-dimensional sets in $\mathbb{R}^N$

That general finite-dimensional sets can be embedded into a Euclidean space of high enough dimension is a result first due to Mañé (1981). The argument here makes use of a powerful extension of this result due to Hunt and Kaloshin (1999) which gives some information on the smoothness of the parametrization of the set that is obtained from this embedding. The statement of the result involves the upper box-counting dimension and the 'thickness' of the set.

The upper box-counting dimension of a set X, measured in a Banach space B, $d(X; B)$, is defined as follows. Let $N_B(X, \epsilon)$ be the minimum number of balls of radius ϵ (in the norm of B) necessary to cover the set X. Then

$$d(X; B) = \limsup_{\epsilon \to 0} \frac{\log N_B(X, \epsilon)}{-\log \epsilon}.$$

This expression essentially captures the exponent d from the relationship $N_B(X, \epsilon) \sim \epsilon^{-d}$. For more on this definition of dimension, see Eden *et al* (1994), Falconer (1990) or Robinson (2001).

If X is a subspace of a Banach space B, then the thickness exponent of X in B, $\tau(X; B)$, is a measure of how well X can be approximated by linear subspaces of B. Denote by $\varepsilon_B(X, n)$ the minimum distance between X and any n-dimensional linear subspace of B. Then

$$\tau(X; B) = \lim_{n \to \infty} \frac{-\log n}{\log \varepsilon_B(X, n)}, \tag{2}$$

which says that if $\varepsilon_B(X, n) \sim n^{-1/\tau}$ then τ is the thickness exponent of X. (Although less elegant, this form of the definition is perhaps more practical than Hunt and Kaloshin's original; the equivalence of the two definitions is shown in lemma 2.1 in Kukavica and Robinson (2004).) Hunt and Kaloshin show that in general $\tau(X; B) \leqslant d(X; B)$.

Theorem 3.1 (Hunt and Kaloshin 1999, theorem 3.6). *Let H be a Hilbert space and $X \subset H$ be a compact set with upper box-counting dimension d and thickness exponent τ (measured in H). Let $N > 2d$ be an integer, and let α be a real number with*

$$0 < \alpha < \frac{N - 2d}{N(1 + \tau/2)}. \tag{3}$$

Then for a prevalent set of bounded linear functions $L : H \to \mathbb{R}^N$ there exists a $C > 0$ such that

$$C|Lx - Ly|^{\alpha} \geqslant |x - y| \qquad \text{for all } x, y \in X. \tag{4}$$

The same result is true if H is a Banach space, but the right-hand side of (3) must be replaced by $(N - 2d)/N(1 + \tau)$.

(The density of Hölder continuous parametrizations of finite-dimensional sets was first shown by Foias and Olson (1996). Hunt and Kaloshin provided the explicit bound on the Hölder exponent in (3) as well as improving 'density' to 'prevalence'.)

They also give an example that shows, in general, that the upper limit on α of $2/(2+\tau)$ is sharp, no matter how large the embedding dimension. Since this upper limit becomes one when $\tau = 0$, it is interesting to have a condition guaranteeing that the thickness is zero (as in theorem 1.2). One such condition is provided by the following result[2] due to Friz and Robinson (1999).

Proposition 3.2. *Let Ω be a bounded subset of $\mathbb{R}^m$. Suppose that X is a subset of $L^2(\Omega)$ that is uniformly bounded in $H^s(\Omega)$. Then $\tau(X; L^2(\Omega)) \leqslant m/s$. In particular, if X consists of 'smooth functions', i.e. is uniformly bounded in $H^s(\Omega)$ for every $s \in \mathbb{N}$, then $\tau(X; L^2(\Omega)) = 0$.*

4. A finite-dimensional delay embedding theorem for Hölder maps

This section gives a statement of a finite-dimensional delay embedding theorem that allows for maps g that are only Hölder continuous. Note, however, that the condition on the map g—that not only g but all its iterates have the same Hölder exponent—is a strong one. Although this is the case for any Lipschitz map g (when $\theta = 1$), it is only true for a subset of all θ-Hölder functions g. Note also that if $\theta = 1$ then the condition on k in the theorem reduces to the familiar $k > 2d$.

Theorem 4.1 (version of theorem 2.7 from Sauer *et al* 1991, allowing for certain Hölder continuous maps). *Let X be a compact subset of $\mathbb{R}^N$ with $d(X) = d$, and $g : X \to X$ a map such that g^r is a θ-Hölder function for any $r \in \mathbb{N}$. Let $k > 2d/\theta$ ($k \in \mathbb{N}$) and assume that the set X_p of p-periodic points of g (i.e. $x \in X$ such that $g^p(x) = x$) satisfies $d(X_p) < p/2\theta$ for all $p = 1, \ldots, k$.*

Let $h_1, \ldots, h_m$ be a basis for the polynomials in N variables of degree at most $2k$, and given any θ-Hölder function $h_0 : \mathbb{R}^N \to \mathbb{R}$ define

$$h_\alpha = h_0 + \sum_{j=1}^{m} \alpha_j h_j.$$

Then the k-fold observation map $F_k : X \to \mathbb{R}^N$ defined by

$$F_k[h_\alpha, g](x) = (h_\alpha(x), h_\alpha(g(x)), \ldots, h_\alpha(g^{k-1}(x)))^T \tag{5}$$

is one-to-one on X for almost every $\alpha \in \mathbb{R}^m$.

(The condition that iterates of g be Hölder is, in fact, only required for $g, \ldots, g^{k-1}$.)

Proof. The proof follows that in Sauer *et al*, apart from minor adjustments in the proof of lemma 4.4/4.5 where the functions $G_0, \ldots, G_t$ are only taken to be θ-Hölder; the image of any ϵ-ball under G_α is then contained in a ball of radius $C\epsilon^\theta$, and $G_\alpha^{-1}(0)$ is empty for almost every α provided that $r > d/\theta$. □

The result in Sauer *et al*'s paper also shows, under conditions on the linearization of g about its periodic orbits, that the observation map F_k is an immersion on all compact subsets of smooth submanifolds of $\mathcal{A}$. It is not clear how to generalize such a result to the case when g is not differentiable, but even if this was possible it would have no implications for the main result given in the next section, since F_k will be 'lifted' to a map on $\mathcal{A}$ via a map which is only Hölder continuous (as in (4)).

[2] There is a small gap in the proof given in this paper, since a function in $H^s(\Omega)$ is not necessarily in $D(A^{s/2})$, where A is the Laplacian on Ω with Dirichlet boundary conditions. This can be corrected by first extending each $u \in X$ to a function in $H^s(\Omega')$ that has compact support in Ω' for some $\Omega' \supset \Omega$, and then considering the Laplacian on Ω'.

5. An infinite-dimensional delay embedding theorem

Theorems 3.1 and 4.1 are now combined to give a topological time delay embedding theorem valid in infinite-dimensional spaces.

Theorem 5.1. *Let $\mathcal{A}$ be a compact subset of a Hilbert space H with upper box-counting dimension $d(\mathcal{A}) = d$, and which has thickness exponent τ. Choose an integer $k > (2+\tau)d$, and suppose further that $\mathcal{A}$ is an invariant set for a Lipschitz map $\Phi : H \to H$, such that the set $\mathcal{A}_p$ of p-periodic points of Φ satisfies $d(\mathcal{A}_p) < p/(2+\tau)$ for $p = 1, \ldots, k$. Then a prevalent set of Lipschitz maps $f : H \to \mathbb{R}$ make the k-fold observation map $D_k[f, \Phi] : H \to \mathbb{R}^k$ defined by*

$$D_k[f,\Phi](u) = (f(u), f(\Phi(u)), \ldots, f(\Phi^{k-1}(u))) \qquad (6)$$

one-to-one on $\mathcal{A}$.

The same result holds if H is replaced by a Banach space B, provided that $k > 2(1+\tau)d$ and $d(\mathcal{A}_p) < p/(2+2\tau)$.

Proof. Given $k > (2+\tau)d$, first choose N large enough that

$$k > \frac{N(2+\tau)}{N-2d}d,$$

and then pick $\alpha < (N-2d)/[N(1+\tau/2)]$ such that $k > 2d/\alpha$.

Use theorem 3.1 to find a bounded linear function $L : H \to \mathbb{R}^N$ that is one-to-one on $\mathcal{A}$ and satisfies

$$c|Lx - Ly|^\alpha \geqslant |x-y| \qquad \text{for all } x, y \in \mathcal{A}.$$

The set $X = L\mathcal{A} \subset \mathbb{R}^N$ is an invariant set for the induced mapping $g : X \to X$ defined by

$$g(\xi) = L\Phi(L^{-1}\xi).$$

Since

$$g^n(\xi) = L\Phi^n(L^{-1}\xi),$$

all the iterates of g are α-Hölder:

$$\begin{aligned}|g^n(\xi) - g^n(\eta)| &= |L\Phi^n(L^{-1}\xi) - L\Phi^n(L^{-1}\eta)| \\ &\leqslant \|L\||\Phi^n(L^{-1}\xi) - \Phi^n(L^{-1}\eta)| \\ &\leqslant l_\Phi^n\|L\||L^{-1}\xi - L^{-1}\eta| \\ &\leqslant c\, l_\Phi^n\|L\||\xi-\eta|^\alpha,\end{aligned}$$

where $\|L\|$ is the operator norm of $L : H \to \mathbb{R}^N$ and l_Φ is the Lipschitz constant of Φ.

Observe that if x is a fixed point of Φ^j then $\xi = Lx$ is a fixed point of g^j, and vice versa. It follows that X_p, the set of all points of X that are p-periodic for g, is given simply by $X_p = L\mathcal{A}_p$. Since L is Lipschitz and the box-counting dimension does not increase under the action of Lipschitz maps[3] (see, e.g., Robinson (2001)), $d(X_p) = d(L\mathcal{A}_p) \leqslant d(\mathcal{E}_p) < p/(2+\tau)$. Similarly, $d(X) \leqslant d(\mathcal{A})$.

Now given a Lipschitz map $f_0 : H \to \mathbb{R}$, define the α-Hölder map $h_0 : \mathbb{R}^N \to \mathbb{R}$ by

$$h_0(\xi) = f_0(L^{-1}\xi) \qquad \text{for all } \xi \in X.$$

[3] Although the dimension cannot increase under the action of Lipschitz maps, it can decrease. It may therefore be possible to improve the result of this theorem by carefully choosing L to ensure that $d(L\mathcal{A})$ is as small as possible. Thanks to David Broomhead for pointing this out.

With $\{h_j\}_{j=1}^m$ as a basis for the polynomials in N variables of degree at most $2k$, all the conditions of theorem 4.1 are satisfied, and hence for almost every $\alpha \in \mathbb{R}^m$, the k-fold observation map on $\mathbb{R}^N$ given by $F_k[h_\alpha, g]$ (see (5)) is one-to-one on X.

Since

$$F_k[h_\alpha, g](\xi) = (h_\alpha(\xi), h_\alpha(g(\xi)), \dots, h_\alpha(g^{k-1}(\xi)))^T,$$

and points in $\mathcal{A}$ and X are in one-to-one correspondence via L ($\xi = Lx$, $x = L^{-1}(\xi)$), the map F_k is equivalent to the k-fold observation map on H given by

$$D_k[f_\alpha, \Phi](x) = (f_\alpha(x), f_\alpha(\Phi(x)), \dots, f_\alpha(\Phi^{k-1}(x)))^T,$$

where $\xi = Lx$ and

$$f_\alpha = h_0 \circ L + \sum_{j=1}^{M} \alpha_j (h_j \circ L) = f_0 + \sum_{j=1}^{M} \alpha_j f_j,$$

where the $\{f_j\}_{j=1}^M$ form a basis for the linear space of polynomials on LH of degree at most $2k$. It follows that a prevalent set of Lipschitz f make the map $D_k[f, \Phi]$ one-to-one on $\mathcal{A}$.

If $\mathcal{A}$ is a subset of a Banach space B, given $k > 2(1+\tau)d$, choose N large enough that

$$k > \frac{2N(1+\tau)}{N_2 d} d,$$

and then pick $\alpha < (N - 2d)/N(1+\tau)$ such that $k > 2d/\alpha$. The argument is then identical to the Hilbert space case. □

Note that the condition on the number of delay coordinates required increases with the thickness of the set $\mathcal{A}$. In the case when $\mathcal{A}$ has zero thickness ($\tau = 0$), this reduces to the $k > 2d$ familiar from the deterministic theory (cf theorem 1.2 in the introduction).

In the case that $\Phi = S(T)$ (the time T map of some underlying continuous time flow), the condition $d(\mathcal{A}_p) < p/(2+\tau)$ precludes the existence of certain periodic orbits. Indeed, for an integer p such that $p/(2+\tau) < 1$, there can be no periodic orbit of period pT, since this would yield a one-dimensional set of p-periodic points for Φ. It follows that the original dynamical system can have no periodic orbits of periods $T, 2T, \dots, p^*T$, where p^* is the largest integer strictly less than $2+\tau$ (or $2+2\tau$ when X is a subset of a Banach space).

It is therefore useful to have a result that prohibits the existence of periodic orbits with small periods. For the finite-dimensional case, Yorke (1969) has shown that any periodic orbit of the ordinary differential equation $\dot{x} = F(x)$ must have period at least $2\pi/L$, where L is the Lipschitz constant of F. In the finite-dimensional case this enables the conditions of theorem 5.1 to be satisfied by taking $\Phi = S(T)$ for T small enough.

That arbitrarily small periodic orbits are prohibited in the Navier–Stokes equations, in not only the two-dimensional but also the three-dimensional case, is shown by Kukavica (1994). Making use of some of his ideas, it is possible to prove a generalization of Yorke's result that is valid for those infinite-dimensional systems that can be written as semilinear evolution equations (Robinson and Vidal-López, 2005).

Let H be a Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$, and let A be an unbounded positive linear operator with compact inverse that acts on H. This means, in particular, that A has a set of orthonormal eigenfunctions $\{w_j\}_{j=1}^\infty$ with corresponding positive eigenvalues λ_j, $Aw_j = \lambda_j w_j$, which form a basis for H. Denote by $D(A^\alpha)$ the domain in H of the fractional power A^α, which in this setting has the simple characterization

$$D(A^\alpha) = \left\{ \sum_{j=1}^\infty c_j w_j : \sum_{j=1}^\infty \lambda_j^{2\alpha} |c_j|^2 < \infty \right\}.$$

Following Henry (1981), consider the semilinear evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + f(u), \tag{7}$$

where $f(u)$ is globally Lipschitz from $D(A^\alpha)$ into H for some $0 \leqslant \alpha \leqslant 1/2$. Then for each α with $0 \leqslant \alpha \leqslant 1/2$ there exists a constant K_α such that if

$$|f(u) - f(v)| \leqslant L|A^\alpha(u - v)| \qquad \text{for all } u, v \in D(A^\alpha),$$

any periodic orbit of (7) must have period at least $K_\alpha L^{-1/(1-\alpha)}$. For such examples it again follows that the condition on the periodic orbits of Φ required by theorem 5.1 can be satisfied by choosing $\Phi = S(T)$ for any T sufficiently small.

6. Conclusion

Theorem 5.1 generalizes the one-to-one portion of the Takens embedding theorem to the infinite-dimensional case, thereby justifying the reconstruction of dynamics from experimental time series in certain spatially extended systems.

A related result, originally proved in the periodic case by Friz and Robinson (2001), and recently generalized by Kukavica and Robinson (2004), shows that a sufficiently large number of point observations are sufficient to distinguish between elements of a finite-dimensional set consisting of *analytic* functions defined on a domain Ω (this can be weakened slightly). If $k \geqslant 16d_f(\mathcal{A}) + 1$ then almost every set $\boldsymbol{x} = (x_1, \ldots, x_k)$ of k points in Ω makes the map

$$u \mapsto (u(x_1), \ldots, u(x_k))$$

one-to-one on X.

What would be more desirable in spatially extended systems such as those modelled by partial differential equations would be to construct a one-to-one time series by sampling at a single spatial point. However, this simple form of result cannot be true in general: consider as in Kukavica and Robinson (2004) the complex Ginzburg–Landau (CGLE),

$$u_t - (1 + \mathrm{i}\nu)u_{xx} + (1 + \mathrm{i}\mu)|u|^2u - au = 0, \tag{8}$$

with periodic boundary conditions on $\Omega = [0, 1]$. If $a > 4\pi^2$ then such a result is not possible. Indeed, given any $x_0 \in \mathbb{R}$, the two explicit solutions

$$u_j(x, t) = \sqrt{a - 4\pi^2}\exp(2\pi\mathrm{i}(-1)^j(x - x_0) - 4\pi^2\mathrm{i}\nu t - a\mu\mathrm{i}t + 4\pi^2\mu\mathrm{i}t),$$

for $j = 1, 2$, which are both contained in the attractor $\mathcal{A}$, coincide at x_0 for all t, while they are clearly distinct. Of course, this does not contradict theorem 5.1, since the set of those observations consisting of point values form a finite-dimensional subset of the Lipschitz observation functions from L^2 into $\mathbb{R}$.

Nevertheless, for this particular example Kukavica and Robinson (2004) have shown that repeated observations at *two* sufficiently close spatial points do serve to distinguish solutions. Note, however, that it cannot be guaranteed that these time points are equally spaced.

Theorem 6.1. *There exists a $\delta_0 > 0$ such that the following holds: let x_1 and x_2 be two points with $|x_1 - x_2| \leqslant \delta_0$, choose $T_0 > 0$, and let $k \geqslant 16d(\mathcal{A}) + 1$. Then for almost every set of k times $\boldsymbol{t} = (t_1, t_2, \ldots, t_k)$ where $t_1, \ldots, t_k \in [0, T_0]$ the mapping $E_t\colon \mathcal{A} \to \mathbb{R}^{2k}$ defined by*

$$E_t(u_0) = (u(x_1, t_1), \ldots, u(x_1, t_k), u(x_2, t_1), \ldots, u(x_2, t_k)),$$

where $u(x, t)$ is the solution with $u(x, 0) = u_0(x)$, is one-to-one on $\mathcal{A}$.

It is shown in the same paper that repeated observations at a single point sufficiently close to the boundary does give a one-to-one mapping for the CGLE with Dirichlet boundary conditions; and that observations at four points that are sufficiently close will work for the Kuramoto–Sivashinksy equation.

It is an outstanding problem to prove a version of the theorem based on measurements at a small number of spatial points repeated at equal time intervals.

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