DIRECT NUMERICAL SIMULATION OF NATURAL CONVECTION OVER HORIZONTAL PLATES

Daniele Venturi¹, Beatrice Pulvirenti¹, Sandro Salvigni¹

¹DIENCA, Universita’ degli Studi di Bologna, Viale Risorgimento 2, 40136 Bologna
e-mail: daniele.venturi@mail.ing.unibo.it

ABSTRACT
Natural convective flows over horizontal plates in confined domains are investigated by direct numerical simulations. A wide range of Rayleigh numbers is considered, spanning steady laminar, unsteady periodic and chaotic transitional regime. The time dependent Boussinesq equations are discretized using a high order spectral/hp finite element method. The averaged Nusselt number turns out to be in very good agreement with the available experimental correlations. Transition form steady to unsteady laminar regime is observed in the range \( Ra = 2.5 \times 10^3 \) to \( Ra = 3.5 \times 10^3 \). The mechanism for heat transfer in proximity of the heated plate is investigated by the symmetric version of the proper orthogonal decomposition (POD) method.

Introduction
We consider the heat transfer and natural convective two dimensional flow over an isothermal upward-facing plate. Natural convective flows induced over horizontal and inclined heated plates have been the subjects of numerous investigations ([8; 7; 15; 6; 19; 18; 2; 10]) in recent decades. These flows are of interest in a number of engineering applications such as computer and electronic cooling, solar collection system and building energy system. We have performed accurate direct numerical simulations of the time dependent Boussinesq equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= \frac{Gr}{Re} \theta j - \nabla p + \frac{1}{Re} \nabla^2 u \\
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta &= \frac{1}{Re Pr} \nabla^2 \theta \\
\nabla \cdot u &= 0
\end{aligned}
\]  

at Rayleigh number ranging from \( Ra = 10^2 \) to \( Ra = 4.75 \times 10^6 \), using the high order spectral/hp finite element method described in [11]. The Prandt number is \( Pr = 0.72 \). Basic details on the numerical scheme used to couple the Navier-Stokes equation to the Fourier equation are given in the Appendix I. Other schemes similar to that proposed here has been proposed in [1] and [13]. In (1) \( j \) is the upward unit vector while \( u \) and \( \theta \) are the dimensionless velocity and temperature fields respectively. The dimensionless temperature is defined by

\[
\theta = \frac{T - T_\infty}{T_w - T_\infty}
\]  

where \( T_w \) is the uniform temperature at the plate surface and \( T_\infty \) is the reference temperature. Also,

\[
Gr = \frac{g \beta (T_w - T_\infty) L^3}{\nu^2}, \quad Pr = \frac{\alpha}{\nu}, \quad Ra = Gr Pr
\]  

Figure 1 shows the computational domain and the boundary conditions which are very similar to those used by Martorell et al. [17]. Kimura et al. [14], Kimura et al. [15] and Martorell et al. [17] et al. performed experiments with horizontal and inclined heated plates with imposed heat flux in similar domains. The flow starts from the rest.
Numerical results

Figure 2 shows some snapshots of the dimensionless temperature field at different Rayleigh numbers. Transition from steady to unsteady laminar convection is observed in the range $Ra = (2.5 \div 3.5) \times 10^3$. In figure 3 we plot the local Nusselt number at the plate surface

$$Nu(x^*, t^*) = \left( \frac{\partial \theta}{\partial y} \right)_{y=0} x^* \in [-0.5, 0.5]$$

(4)

for the snapshots shown in figure 2.

In order to compare the heat transfer’s features of plates of various shapes Goldstein et al [9] proposed a length-scale $L$ defined as the active surface divided by its perimeter. For our infinite horizontal strip we have

$$L = \lim_{b \to \infty} \frac{ab}{2(a+b)} = \frac{a}{2},$$

(5)

where $a$ denotes the width of the plate. Many experimental correlations $Nu - Ra$ such as those given by Al-Arabi & El-Riedy [2] and Bandrowski & Rybski [5], originally based upon the width of the plate $a$, have been recalculated by Goldstein & Kei-Shun [8] as function of the area to perimeter length-scale $L$. In figure 4 we plot the correlations proposed in [2],[16],[5] and [9] for the dependence of the averaged Nusselt number on the Rayleigh number when these dimensionless groups are based on the area to perimeter ratio (see also [17]). Table 1 summarizes the aforementioned published experimental correlations and their range of applicability.

<table>
<thead>
<tr>
<th>Correlation</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>Range of $Ra_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goldstein et al. [9]</td>
<td>0.96</td>
<td>0.167</td>
<td>$1 \div 2 \times 10^2$</td>
</tr>
<tr>
<td>“”</td>
<td>0.59</td>
<td>0.25</td>
<td>$2 \times 10^2 \div 2 \times 10^4$</td>
</tr>
<tr>
<td>Bandrowski et al. [5]</td>
<td>0.69</td>
<td>0.222</td>
<td>$3 \times 10^3 \div 2.5 \times 10^5$</td>
</tr>
<tr>
<td>Lloyd et al. [16]</td>
<td>0.54</td>
<td>0.25</td>
<td>$2 \times 10^4 \div 8 \times 10^6$</td>
</tr>
<tr>
<td>Al-Arabi et al. [2]</td>
<td>0.50</td>
<td>0.25</td>
<td>$2 \times 10^5 \div 8 \times 10^6$</td>
</tr>
</tbody>
</table>

Table 1. Experimental correlations $Nu_L = c_1Ra_L^{c_2}$ for isothermal horizontal plates. The characteristic length is the active surface divided by its perimeter.
we extracted 50 equidistant snapshots of the dimensionless temperature using a convenient sampling frequency in order to satisfy the Nyquist’s criterion. The averaged Nusselt number

\[ Nu_L = \frac{1}{t} \int_{0}^{t} \int_{-0.5}^{0.5} Nu(x^*, t^*) \, dx^* \, dt^*, \]

where \( t \) is period of observation, turns out to be in very good agreement with the experimental correlations (see figure 4). This supports the accuracy of the numerical scheme proposed in Appendix I. Table 2 summarizes the main findings for the averaged Nusselt number.

**POD analysis of the temperature field at \( Ra = 4.75 \times 10^6 \)**

We consider the dimensionless flow at \( Pr = 0.72 \) and \( Ra = 4.75 \times 10^6 \), the latter being based upon the width of the plate \( a \). This flow corresponds to the physical case of natural convection over an isothermal highly elongated plate of width \( a = 0.1 \) m at \( T_w = 400 \) K placed on an insulator of width 1 m inside a box of air of width 1 m and height 0.7 m with walls kept at \( T_\infty = 300 \) K. If we evaluate the physical properties of the air at the mean film temperature \( T_f = (T_w + T_\infty) / 2 = 350 \) K we get \( Pr = 0.719 \) and \( Gr \approx 6.60 \times 10^6 \), i.e. \( Ra \approx 4.75 \times 10^6 \). A Fourier analysis of the time series for the velocity (figure 6) and the temperature fields reveals that the highest dimensionless time frequency is everywhere less than \( f_{sup} = 6 \). We extracted \( S = 65 \) regular spaced snapshots from the DNS using a dimensionless sampling frequency \( f = 16 > 2f_{sup} \) in order to satisfy the Nyquist’s criterion. The dimensionless frequency 0.246 characterizes the most energetic dimensionless vortex shedding close to the heated plate (see figure 6).

We investigate the dimensionless temperature field by the symmetric version of the proper orthogonal decomposition method [4; 3]. Essentially this consists in representing the dimensionless temperature by a biorthogonal series

\[ \theta(x, t) = \sum_{k=1}^{\infty} \sqrt{\mu_k} \Phi_k(x) \psi_k(t) \]

where the functions \( \psi_k \) and \( \Phi_k \) are the solutions of the following

\[ \text{Rayleigh number } Ra \quad \text{Averaged Nusselt number } Nu_L \]

\begin{tabular}{|c|c|}
\hline
\hline
\text{Rayleigh number } Ra & \text{Averaged Nusselt number } Nu_L \\
\hline
10^2 & 1.96 \\
5 \times 10^2 & 2.84 \\
10^3 & 3.34 \\
5 \times 10^3 & 4.79 \\
10^4 & 5.59 \\
5 \times 10^4 & 8.15 \\
10^5 & 9.87 \\
5 \times 10^5 & 14.66 \\
10^6 & 16.23 \\
4.75 \times 10^6 & 25.98 \\
\hline
\end{tabular}
The POD modes \( \psi_k \) and \( \Phi_k \) are not independent as there exist a one-to-one correspondence between them in the form

\[
\langle \psi_k \rangle (t) \equiv \int_\Omega \theta (x, t) \cdot \psi_k (x) \, dx = \sqrt{\mu_k} \psi_k (t) , \tag{12}
\]

\[
\langle \phi_k \rangle (x) \equiv \int_T \theta (x, t) \psi_k (t) \, dt = \sqrt{\mu_k} \Phi_k (x) . \tag{13}
\]

These relations, called dispersion relations (see [4],[3]), characterize the link between the spatial and the temporal evolution of the system. The POD analysis of the temperature field at \( Ra = 4.75 \times 10^6 \) is shown in figure 7. The first temperature mode represents the time averaged temperature which carries on

\[
\text{Figure 5. Dimensionless temperature field at } Ra = 4.75 \times 10^6 .
\]

\[
\text{Figure 6. Fourier spectrum of the time series for the y (a) and x (b) velocity components at the point } x^* = -0.1, y^* = 0.23 .
\]

the 96% of the total energy (see figure 7(a)). This justifies the steady state numerical (or analytical) solutions for natural convection above horizontal plates and the usage of low frequency acquisition systems to measure the Nusselt number. The time averaged (or mean) heat flux at the plate surface can be written as

\[
\langle q \rangle_t = \frac{\kappa (T_w - T_\infty)}{L} \left. \frac{\partial \langle \theta \rangle_t}{\partial y^*} \right|_{y=0} \tag{14}
\]

\[
= \mu_1 \kappa \frac{(T_w - T_\infty)}{L \sqrt{t}} \left. \frac{\partial \Phi_1}{\partial y^*} \right|_{y=0} \tag{15}
\]

(see figure 7(b)) where \( \kappa \) is the thermal conductivity of the fluid and \( \langle \cdot \rangle_t = \frac{1}{T} \int_0^T \cdot \, dt \) is the averaging operator. Let us approximate the heat flux at the plate surface by a third order temperature POD expansion

\[
q \approx \langle q \rangle_t + \frac{\kappa (T_w - T_\infty)}{L} \left( \mu_2 \psi_2 \left( \frac{t}{\tau} \right) \frac{\partial \Phi_2}{\partial y^*} \right|_{y=0} + (16)
\]

\[
+ \mu_3 \psi_3 \left( \frac{t}{\tau} \right) \frac{\partial \Phi_3}{\partial y^*} \right|_{y=0} \tag{17}
\]

where \( \tau \) is the time scale. From the structure of the second and the third temperature POD modes shown in figure 7(c)(d) (spatial modes) and figure 8 (temporal modes) respectively, we can draw conclusions on the main mechanism responsible for the heat transfer. In fact the symmetric-like structure of \( \Phi_2 \) and \( \Phi_3 \) and the periodic-like behavior of the corresponding temporal modes (figure 8) implies that the heat flux at the plate surface is waveform-like in time and symmetric in space, 0.246 being the main dimensionless time frequency for both the aforementioned temperature modes. Such frequency is in very good agreement with the main vortex shedding time frequency. This supports the conclusion that this temperature coherent structure is determined by advection. Therefore the most energetic temporal evolution of the chaotic heat transfer is characterized by the source zones highlighted in figure 7(c)-(d) moving on the plate surface symmetrically and periodically from the center to the plate edges.

Summary

We performed direct numerical simulations of natural convective flows over isothermal horizontal plates in confined domains. We investigated a wide range of Rayleigh numbers from
$Ra = 10^2$ to $Ra = 4.75 \times 10^6$. Transition form steady to unsteady state is observed in the interval $Ra = (2.5 \div 3.5) \times 10^4$. The computed correlation between the averaged Nusselt number and the Rayleigh number turns out to be in very good agreement with the experimental correlations. The proper orthogonal decomposition of the temperature field at Rayleigh number $Ra = 4.75 \times 10^6$ reveals the existence of organized spatio-temporal structures which characterize the evolution of the main heat transfer’s features at the plate surface.

**APPENDIX I**

**Basic details of the numerical scheme**

To discretize the equations (1) we use the high order splitting scheme described in [11] both for Fourier and Navier-Stokes equations. The splitting scheme, involves the following substeps:

$$\bar{\theta} = \sum_{q=0}^{J_1-1} \alpha_q \theta^{n-q} + \Delta t \sum_{q=0}^{J_2-1} \beta_q [u \cdot \nabla \theta]^{n-q}$$

(18)
\[
\left( \nabla^2 - \frac{\gamma_0 \text{RePr}}{\Delta t} \right) \theta^{n+1} = -\frac{\text{RePr} \lambda}{\Delta t} \theta^n
\]  
(19)

\[
\tilde{u} = \sum_{q=0}^{J_u-1} \alpha_q u^{n-q} + \Delta t \left[ \sum_{q=0}^{J_u-1} \beta_q \left( \left[ (u \cdot \nabla) u \right]^{n-q} + \frac{\mathcal{G}_r}{\text{Re}} \theta^{n-q+1} \right) \right]
\]
(20)

\[
\nabla \tilde{p}^{n+\frac{1}{2}} = \nabla \cdot \left( \frac{\tilde{u}}{\Delta t} \right)
\]  
(21)

\[
\left( \nabla^2 - \frac{\gamma_0 \text{RePr}}{\Delta t} \right) u^{n+1} = \frac{\text{RePr}}{\Delta t} \tilde{u} - \text{RePr} \nabla \tilde{p}^{n+\frac{1}{2}}
\]  
(22)

where \( J_u, J_t \) are the order of integration (we use \( J_u = J_t = 3 \)) for the advection and the diffusion terms respectively (stiffly stable SE/SI scheme [11]). For the spatial discretization we use an 8th order spectral finite element method (see [11] and [12]). Also, \( u^{n+1} \) denotes the velocity field at the iteration \( n+1 \). As usual in the spectral projection method, the divergence free constraint for the velocity field is enforced in the fourth sub-step, i.e. equation (21).

**NOMENCLATURE**

- \( a \) Width of the plate
- \( Gr \) Grashof Number
- \( \kappa \) Fluid thermal conductivity
- \( L \) Area to perimeter length scale
- \( Nu \) Local Nusselt number
- \( Nu_L \) Averaged Nusselt number
- \( p \) Dimensionless piezometric pressure
- \( Pr \) Prandtl Number
- \( q \) Heat flux at the plate surface
- \( Ra \) Rayleigh Number
- \( Re \) Reynolds Number
- \( S \) Spatial autocorrelation
- \( t \) Dimensionless period of observation
- \( T \) Temporal autocorrelation
- \( u \) Dimensionless velocity field
- \( x^* \) Dimensionless horizontal coordinate
- \( y^* \) Dimensionless vertical coordinate

**Greek symbols**

- \( \alpha_q \) Coefficients for Adams-Bashforth integration scheme
- \( \beta_q \) Coefficients for Adams-Bashforth integration scheme
- \( \Phi_k \) POD spatial modes
- \( \mu_k \) POD eigenvalues (energy)
- \( \psi_k \) POD temporal modes
- \( \theta \) Dimensionless temperature field
- \( \Theta \) Integral operator with kernel \( \theta \)
- \( \Theta^\dagger \) Adjoint of \( \Theta \)
- \( \tau \) Time scale

**REFERENCES**


Second National Conference on Computational Mechanics, Trondheim.


